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Q-function measurement by projection synthesis

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Abstract

Following a recent proposal by Barnett and Pegg [Phys. Rev. Lett. 76 (1996) 4148] we present an extension of the projection synthesis method for the Q-function measurement. We also show that a convenient choice of a reference state allows us to measure dispersions of quadrature operators. © 1997 Elsevier Science B.V.

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In a recent paper Barnett and Pegg [1] presented an interesting proposal for measuring the phase of a field state $|\psi\rangle$ by experimentally determining its probability density $p_\psi(\theta) = |\langle\theta|\psi\rangle|^2$, where $|\theta\rangle$ is the truncated phase state

$$|\theta\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{in\theta} |n\rangle, \tag{1}$$

which is an approximation to the phase state introduced by the same authors [2]. Hence,

$$p_\psi(\theta) = \frac{1}{2\pi} \left| \sum_{n=0}^{N} C_n e^{-in\theta} \right|^2 = \langle\psi|\hat{\pi}|\psi\rangle, \tag{2}$$

where $C_n = \langle n|\psi\rangle$ and $\hat{\pi}$ is the phase projector (K is a positive constant)

$$\hat{\pi} = K|\theta\rangle\langle\theta|. \tag{3}$$

If the field is in a mixed state $\hat{\rho}$, then Eq. (2) is modified to the form (up to a constant)

$$P_\rho(\theta) = \mathrm{tr}(\hat{\rho}\hat{\pi}) = \langle\theta|\hat{\rho}|\theta\rangle, \tag{4}$$

where Eq. (3) was employed.

The experimental arrangement of Ref. [1] is reproduced in Fig. 1: a field mode in a state $|\psi\rangle_a$ is coherently mixed in a beam splitter with a reference field mode b in a state $|B\rangle_b$. The authors in Ref. [1] showed that specifying a suitable reference state $|B\rangle_b$ and the photocounting in the two outputs of the beam splitter leads to the required probability distribution (2) or (4). In this way, measuring the probability of finding n_a and n'_b photons in the output modes a and b, respectively, one obtains $p(n_a, n'_b) = {}_a\langle\psi|\hat{\pi}|\psi\rangle_a$, where $\hat{\pi}$ is given by the projector

$$\hat{\pi} = {}_b|B|\hat{R}^+|n'\rangle_b|n\rangle_a\,{}_a\langle n|{}_b\langle n|\hat{R}|B\rangle_b \tag{5}$$

and $\hat{R}$ is a (unitary) transformation [3] linking the

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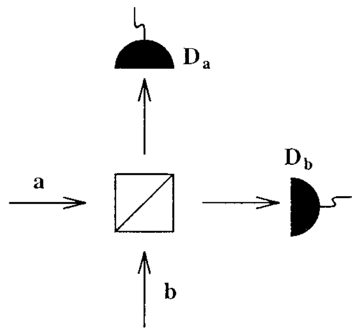

Fig. 1. Sketch of the experimental setup with a beam splitter, input modes a and b, and detectors D_a and D_b.

output modes to input modes. For a 50/50 beam splitter $\hat{R}$ is given by

$$\hat{R} = e^{i(\pi/4)(ab^+ + a^+b)}$$
$$= e^{ib^+a}[e^{\eta(\hat{n}_b - \hat{n}_a)}]e^{iba^+}, \qquad (6)$$

with $\eta = (\ln 2)/2$.

In Ref. [1] it was shown that for $n_a = N$ and $n'_b = 0$, corresponding to N photocounts in the detector D_a and no counts in D_b, the reference state $|B\rangle_b$ yielding the phase-projector $\hat{\pi} = K|\theta\rangle\langle\theta|$, given in Eq. (3), is

$$|B\rangle_b = C\sum_{k=0}^{N}\binom{N}{k}^{-1/2} e^{ik(\theta-\pi/2)}|k\rangle_b, \qquad (7)$$

which was called reciprocal-binomial state, due to the presence of the binomial factor $\binom{N}{n}^{-1/2}$, the reciprocal of that in the binomial state introduced by Stoler et al. [4]. C is a normalization constant.

In the present paper we will study the conditions that allow one to get the measurement of the Q-distribution function [5] through the projection synthesis scheme introduced in Ref. [1]. This procedure would be alternative to others in the literature, using the method of quantum tomography [6], or the method of eight-port homodyning detection [7].

Next, if we have that

$$\hat{\pi} = K|\alpha\rangle\langle\alpha| \qquad (8)$$

its substitution in the first equality of Eq. (4) gives

$$\rho(\alpha) = \mathrm{tr}(\hat{\rho}\hat{\pi}) = \langle\alpha|\hat{\rho}|\alpha\rangle = Q(\alpha), \qquad (9)$$

where $Q(\alpha)$ is the Q-distribution function, by definition [5]. Hence, the point to be solved here is: in what fashion we might choose n_a, n'_b and the reference state $|B\rangle_b$ in such a way that the synthesized projector $\hat{\pi}'$,

$$\hat{\pi}' = {}_b\langle B|\hat{R}^+|n\rangle_a|n'\rangle_b\,{}_b\langle n'|{}_a\langle n|\hat{R}|B\rangle_b \qquad (10)$$

will coincide with that in Eq. (8), namely, $\hat{\pi}' = \hat{\pi} = K|\alpha\rangle\langle\alpha|$?

If we set $n_a = N$, $n'_b = 0$, as in Ref. [1], then the required form of the reference state $|B\rangle_b$ leading to the projector (8) is

$$|B\rangle_b = C\sum_{k=0}^{N} b_k|k\rangle_b, \qquad (11)$$

where C is a normalization constant and

$$b_k = \frac{2^{-N}}{C}\binom{N}{k}^{-1/2} e^{ik\pi/2}C^*_{N-k}. \qquad (12)$$

The truncated state (11) will lead, in fact, to a truncated coherent state projector. However, the procedure stands as a good approximation under specified conditions as discussed below. The present approximation for the coherent state projector is the analogy of the approximation in Ref. [1] for the phase state projector, as we have mentioned before. The coefficient $C_{N-k} = \langle N-k|\alpha\rangle$ should be identified with the $(N-k)$th coefficient of the coherent state in the number representation, namely,

$$C_{N-k} = e^{-|\alpha|^2/2}\frac{\alpha^{N-k}}{\sqrt{(N-k)!}}, \qquad (13)$$

with $k = 0, 1, 2, \dots, N$. Note that $C_{N-k} = 0$ for $k > N$.

From Eqs. (12) and (13) it is natural to refer to these states as complementary-coherent states (CCS), by analogy with the coherent states of Glauber [8], since the coefficients b_k of the CCS are connected with the coefficients C_{N-k} of the coherent states.

At this point a clarification concerning the number N deserves attention: in the scheme of Barnett and Pegg, N is a well-defined number entering the definition of the phase state onto which they intended to make projections. However, at first sight it is apparent that in our scheme N has no natural definition, seeming to be a number which can be chosen arbitrarily. This appearance emerges from the fact that the identification in (12) imposes that C_{n-k}

$= 0$ for $k > N$, which leads to $b_k = 0$ for $k > N$. Hence our CCS is a truncated state, in this respect being similar to the reciprocal-binomial state of Barnett and Pegg [1]. Thus it seems that this correspondence between a truncated state (the CCS) and the coherent state is misleading, since in the latter $c_k \neq 0$ for $k > N$. Hence, what we have really done here is devise a scheme for making projections onto *truncated* coherent states. As a consequence, it could be argued that we do not strictly measure the Q-function, which is obtained from $\langle \alpha | \hat{\rho} | \alpha \rangle$ where $| \alpha \rangle$ is a coherent state. Notwithstanding, a good approximation can be achieved in most situations if N is sufficiently large, in the sense that $N \gg |\alpha|^2$, where $|\alpha|^2 = \bar{n}_\alpha$ is the average excitation of the synthesized coherent state. In other words, the approximation

$$| \alpha \rangle = \sum_{n=0}^{\infty} c_n | n \rangle \cong \sum_{n=0}^{N} c_n | n \rangle$$

is a good approximation if $N \gg |\alpha|^2$. In practice, this condition will restrict the average excitation of the a-mode to small values, i.e. ${}_a\langle \psi | \hat{n} | \psi \rangle_a \ll N$ or, alternatively, $(\bar{n}_\alpha = \mathrm{tr}(\hat{\rho}_a \hat{n}) \ll N$.

The physical justification of the present work lies in the new scheme furnishing a good approximation for the Q-function measurement, which is alternative to that using quantum tomography [6]. As in Ref. [1], the crucial point to be implemented here is the preparation of the required reference state, the CCS. Following the words of Barnett and Pegg [1], in light of the recent works [8], we hope that the problem of generation of specific states, as the CCS, will be solved in future. In fact, a feasible proposal for the preparation of the reciprocal-binomial state, employed in Ref. [1], was presented recently [9]. Concerning the CCS, a proposal for its preparation remains to be done.

Taking advantage of the present strategy, one can alternatively measure the variances $\Delta \hat{x}_i^2$, $i = 1, 2$, of quadrature operators [5], $\hat{x}_1 = (\hat{a} + \hat{a}^+)/2$, $\hat{x}_2 = (\hat{a} - \hat{a}^+)/2\mathrm{i}$. In this case the pertinent projector to be constructed is

$$\hat{\pi}_i = K | x_i \rangle \langle x_i | \tag{14}$$

to obtain the probability density (compare with (2))

$$p_\psi(x_i) = \langle \psi | \hat{\pi}_i | \psi \rangle = \langle \psi | x_i \rangle \langle x_i | \psi \rangle = | \langle x_i | \psi \rangle |^2 \tag{15}$$

of finding the quadrature x_i. The variance $\Delta \hat{x}_i^2$ will be obtained from

$$\Delta \hat{x}_i^2 = \langle \hat{x}_i^2 \rangle - \langle \hat{x}_i \rangle^2 = \int x_i^2 p(x_i)\, \mathrm{d}x_i - \left[\int x_i p(x_i)\, \mathrm{d}x_i \right]^2. \tag{16}$$

To this end, we must synthesize the projector $\hat{\pi}_i$ in Eq. (14), for a selected quadrature, $\hat{x}_1$ or $\hat{x}_2$. To find the appropriate reference state $| B \rangle_b$ we set

$$\hat{\pi}_1 = \langle x_1 \rangle \langle x_1 | = {}_b\langle B | \hat{R}^+ | n \rangle_a | n' \rangle_b \, {}_b\langle n' | {}_a\langle n | \hat{R} | B \rangle_b, \tag{17}$$

with

$$| B \rangle_b = C \sum_k B_k | k \rangle_b, \tag{18}$$

where C is a normalization constant. Trying again $n_a = N$, $n'_b = 0$ we obtain, with $\eta = (\ln 2)/2$,

$${}_a\langle N | {}_b\langle 0 | \hat{R} | B \rangle_b = \mathrm{e}^{-\eta N} \sum_{n=0}^{\infty} \sum_k {}_a\langle N | {}_b\langle 0 | \frac{\mathrm{i}^n (a^+)^n b^n}{n!} | k \rangle_b = \sum_{n=0}^{N} \mathrm{e}^{-\eta N} \binom{N}{n}^{1/2} \mathrm{e}^{\mathrm{i} n \pi/2} B_n \, {}_a\langle N - n |. \tag{19}$$

Next we identify

$$| x_1 \rangle = C^* \sum_{n=0}^{N} \mathrm{e}^{-\eta N} \mathrm{e}^{\mathrm{i} n \pi/2} \binom{N}{n}^{1/2} B_n^* \, {}_a| N - m \rangle \tag{20}$$

to find

$$\langle m | x_1 \rangle = C_m(x_1) = \mathrm{e}^{-\eta N} k \mathrm{e}^{\mathrm{i}(N-m)\pi/2} \binom{N}{m} B_m^*. \tag{21}$$

The coefficients $C_m(x_1)$ are given by [10]

$$C_m(x_1) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{(2m-1)!!}} H_m(x_1/\sqrt{2}) \mathrm{e}^{-x_1^2/2}, \tag{22}$$

where $H_m(\xi)$ is the Hermite polynomial. From Eq. (21) we obtain the coefficients of our wanted reference state

$$B_k = \binom{N}{k}^{-1/2} e^{-\eta N} e^{ik\pi/2} C_{n-k}(x_1), \tag{23}$$

which synthesizes the projector $\hat{\pi}_1 = |x_1\rangle\langle x_1|$. A similar procedure allows one to get $p_\psi(x_2) = |\langle x_2|\psi\rangle|^2$, hence the variance $\Delta\hat{x}_2^2$ (cf. Eq. (16)).

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