17 April 2000

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 268 (2000) 260–267

www.elsevier.nl/locate/physleta

On the generation of the phase state

Y. Guimarães [a], B. Baseia [a], C.J. Villas-Boas [b], M.H.Y. Moussa [b,*]

[a] *Instituto de Física, Universidade Federal de Goiás, Caixa Postal 131, 74001-970, Goiânia (GO), Brazil*
[b] *Departamento de Física, Universidade Federal de São Carlos, Via Washington Luiz, Km 235, São Carlos, 13565-905 (SP), Brazil*

Received 20 December 1999; accepted 3 March 2000
Communicated by P.R. Holland

Abstract

Proposals for the generation of the phase state are presented, for both stationary and travelling fields. In the first case the scheme combines two variants of the 'standard' cavity QED quantum state engineering. The second scheme consists in an extension of the optical state truncation by projection synthesis [Pegg et al., Phys. Rev. Lett. 81 (1998) 1604], which is based on field mixing in a beam-splitter array. The natural restriction to realize an *exact* phase state is discussed. © 2000 Published by Elsevier Science B.V. All rights reserved.

PACS: 42.50.Dv; 03.65.Db; 03.65.Bz

1. Introduction

Among the various interesting states of quantum optics studied nowadays, the phase state $|\theta\rangle$ is one very representative in this class [1–3]. According to a theorem by Hillery [4], this pure state belongs to the class of nonclassical states. It is complementary to the number state $|n\rangle$, in the sense that the number operator $\hat{n}$ ($\hat{n}|n\rangle = n|n\rangle$) and the phase operator $\hat{\phi}$ ($\hat{\phi}|\theta\rangle = \theta|\theta\rangle$) form a canonically conjugate pair. In certain sense, the coherent state $|\alpha\rangle$ ($\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\hat{a}$ being the annihilation operator) lies between them: it exhibits phase and number dispersions, $\Delta\hat{\phi}$ and $\Delta\hat{n}$, between those values found in the phase state and in the number state [5].

Defining a phase state is a long story, because of the difficulties associated with the definition of a Hermitean phase operator [6–8]. Several descriptions of phase operators, based on different assumptions and mathematical constructions, can be found in the literature, the first consistent description of a phase operator (and a phase state) being presented by Susskind and Glogower [9]. New phase operators and phase states have been introduced, as we can see in a review by Carruthers and Nieto [10]. The more recent proposal of a Hermitean phase operator (and phase state) was introduced by Pegg and Barnett [1–3]. They have shown that a finite-dimensional phase operator (the phase state belonging to a truncated Hilbert space) is better behaved than the Susskind–Glogower counterpart, for example leading $\langle\cos^2\hat{\phi}\rangle + \langle\sin^2\hat{\phi}\rangle$ to fulfill the standard trigonometric property of unit rule. The apparent success of the Pegg–Barnett phase state, however, is not con-

* Corresponding author.
E-mail addresses: yara@fis.ufg.br (Y. Guimarães), miled@power.ufscar.br (M.H.Y. Moussa).

PII: S0375-9601(00)00168-7

sensual and the reader is referred to [11–13] for controversies.

Now, having obtained a satisfactory definition of a phase operator and phase state, namely [1–3],

$$\hat{\phi} = \sum_m \theta_m |\theta_m\rangle\langle\theta_m|, \tag{1}$$

and

$$|\theta_m\rangle = \frac{1}{\sqrt{1+N}} \sum_{k=0}^{N} e^{ik\theta_m} |k\rangle, \tag{2}$$

with θ_m satisfying appropriate requirements shown in Ref. [1–3], a new problem emerges concerning with how to generate this interesting state. Actually, as in the case of plane waves, the phase state involves infinite energy [5]. Hence, strictly speaking, it cannot be generated exactly in practice. However, approximations to this state is not forbidden by restricting ourselves to a truncate Hilbert space. To our knowledge, a discussion about the generation of a phase state and the way in which its restrictions come out in practice, is a subject of very recent investigation, as mentioned below.

With respect to propagation in space, there are two classes of field states[1]: one of them belonging to the class of *stationary-waves* (trapped fields inside high-Q cavities); the other concerning with the class of *travelling-waves*. The latter occurs in natural sources (thermal light) and, in general, seems easier to be realized in laboratories (chaotic and laser light being examples of such category). Usually, the squeezed state is generated in a travelling-wave [14–16], this being also the case of the first proposal for generation of a Schrödinger's cat state [17]. Here, it is worth mentioning recent (ingenious) proposals by Pegg et al. [18] where a running-wave is generated in the truncated state, $c_0|0\rangle + c_1|1\rangle$, and by Dakna et al. [19] where the generation of arbitrary quantum states of travelling fields is studied. On the other hand, the development of superconducting high-Q cavities in microwave domain [20] opened new and interesting research in quantum optics. In this way, coherent states using classical sources [20], squeezed states (see, e.g., Ref. [21]) and Schrödinger's cat states [22,23] can be generated in stationary waves, inside high-Q cavities. Depending on which case we are interested in, two different approaches have been employed: (i) to make a quantum measurement (selective detection) on one part of an entangled system to obtain the other part projected onto the desired state. This procedure, called 'state reduction', involves *non-unitary* evolution of the field state; (ii) to find an appropriate Hamiltonian describing a nonlinear medium which leads a given initial state to the desired state via *unitary*evolution. Usually the case (i) (case (ii)) is concerned with stationary (travelling) waves. However, selective detection leading to state reduction (non-unitary evolution) has also been employed recently in the generation of states of travelling fields [18,19].

In the present work we will discuss a proposal to generate a phase state and discuss its intrinsic limitation [5] in experiments. One proposal concerns with the case of stationary fields, discussed in Section 2. The other concerns with travelling fields, discussed in Section 3. Section 4 contains the comments and conclusion.

2. Generation of phase state for *stationary* fields

Here, as done in Ref. [24], we will combine two schemes used in 'standard' cavity QED quantum state engineering (QSE) [25–27], namely, those using resonant and non-resonant (dispersive) atom-field interactions.

Fig. 1 shows the experimental arrangement consisting of N identical two- level atoms (levels $|g\rangle$ and $|e\rangle$), a Ramsey zone R, an initially empty high-Q cavity C, and a state-selective field ionization detector D. The two-level atoms are considered as

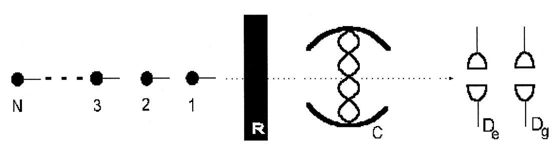

Fig. 1. Sketch of the experimental setup for generation of phase state for *stationary* waves. The symbols R, C, and D stand for the Ramsey zone, the cavity, and the detector chamber, respectively. The numbers 1,2, ..., N, indicate the required atoms.

[1] This scenario goes beyond the realm of quantum optics: in quantum mechanics it appears with names bound and unbound states. In the first (second) case we are faced with a discrete (continuous) spectra, the particle in a box and the free particle being trivial examples of them, respectively.

circular Rydberg atoms, suitable for preparing atom-field long-lived correlations [28,29]. They present a strong coupling to microwaves and a very long radiative decay time of the order of cavity lifetimes for high-Q superconducting cavities. Moreover, Rydberg atomic states are counted with high efficiency by state selective field ionization detectors. Each atom is prepared by the microwave field R in a given $|\mathbf{g}\rangle$, $|\mathbf{e}\rangle$, superposition state $C_g^{(j)}|g\rangle_j + C_e^{(j)}|e\rangle_j$ (j labelling the jth atom) at a given time with a well-defined velocity. The atoms are thus sent, one-by-one, through cavity C, and after their on-resonant interaction with the field in C, they are counted in D. The on-resonant interaction is described by the Jaynes–Cummings Hamiltonian $H_{on} = \hbar\Omega_1 \cdot (\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)$, where $\hat{\sigma}_+ = |e\rangle\langle g|$, $\hat{\sigma}_- = |g\rangle\langle e|$, and Ω_1 is the one-photon Raby frequency. Once the N atoms have crossed the cavity, a straightforward calculation allows us to verify that when all of them are detected in their ground state, leaving their photon in the cavity, the pure field state of the cavity results [24]

$$|\phi\rangle = \frac{1}{\mathcal{N}} \sum_{k=0}^{N} \Lambda_k^N e^{-ik\pi/2} |k\rangle, \tag{3}$$

where $\mathcal{N}$ is a normalization constant and the coefficients Λ_k^N are given by the recurrence formula

$$\Lambda_k^N = (1-\delta_{k,0})\,\Lambda_{k-1}^{N-1} c_e^N \sin\left(\sqrt{k}\,\Omega_1\tau_N\right) + (1-\delta_{k,N})\,\Lambda_k^{N-1} c_g^N \cos\left(\sqrt{k}\,\Omega_1\tau_N\right), \tag{4}$$

with $\Lambda_0^0 = 1$. The probability for detecting the whole series of atoms in the ground state is of the order of $1/2^N$, restricting the technique for N not too large.

Now, as a preliminary strategy we employ the method above to generate the truncated state ($\mathcal{N}'$ standing for normalization)

$$|\Psi\rangle = \frac{1}{\mathcal{N}'} \sum_{k=0}^{N} |k\rangle. \tag{5}$$

To this end we identify (3) [plus (4)] with (5), i.e., setting $|\phi\rangle = |\Psi\rangle$ we obtain

$$\Lambda_k^N = e^{ik\pi/2} = (i)^k. \tag{6}$$

As example, for $N=2$ the application of the recurrence formula (4) to the Eq. (6) gives, step by step,

$$k=0: \Lambda_0^{(2)} = C_g^{(1)} C_g^{(2)} = 1, \tag{7a}$$

$$k=1: \Lambda_1^{(2)} = C_g^{(1)} C_e^{(2)} \sin(\Omega\tau_2) + C_g^{(2)} C_e^{(1)} \sin(\Omega\tau_1)\cos(\Omega\tau_2) = i, \tag{7b}$$

$$k=2: \Lambda_2^{(2)} = C_e^{(1)} C_e^{(2)} \sin(\Omega\tau_1)\sin\left(\sqrt{2}\,\Omega\tau_2\right) = -1. \tag{7c}$$

Dividing (7b) and (7c) by (7a) furnishes a coupled set of algebraic equations for the parameter $\xi^{(j)} = C_e^{(j)}/C_g^{(j)}$, $j=1,2$. Hence, Eq. (6) [plus (4)] allows one to obtain the relation $\xi^{(j)} = C_e^{(j)}/C_g^{(j)}$ for the jth atom, entering the cavity. Now, setting $C_e^{(j)} = ic_e^{(j)}$, $C_g^{(j)} = c_g^{(j)}$, we obtain

$$\frac{c_e^{(1)}}{c_g^{(1)}} = \frac{2\sin(\Omega\tau_2)}{\sin(\Omega\tau_1)\sin\left(\sqrt{2}\,\Omega\tau_2\right)}\eta^{-1}(\tau_2), \tag{8a}$$

$$\frac{c_g^{(2)}}{c_g^{(2)}} = \frac{1}{2\sin(\Omega\tau_2)}\eta(\tau_2), \tag{8b}$$

where

$$\eta(\tau_2) = 1 + \left[1 - \frac{2\sin(2\Omega\tau_2)}{\sin\left(\sqrt{2}\,\Omega\tau_2\right)}\right]^{\frac{1}{2}}. \tag{9}$$

The knowledge of $\xi^{(j)}$ gives the superposition state $|\Psi_{(0)}\rangle_A = C_g^{(j)}|g\rangle_j + C_e^{(j)}|e\rangle_j$ in which we must prepare the jth atom entering the cavity. It is found that the variable $\xi^{(j)}$ obeys an algebraic equation of degree N. For $N \le 4$ there are algorithms yielding analytical solutions, whereas for $N > 4$, no such algorithms exist, in general: only numerical solutions, coming from a set of $N+1$ coupled equations, are available in this case.

Next, after obtaining the state (5) an auxiliary atom is immediately sent across the cavity, tuned to have an off-resonant (dispersive) interaction with the field in the state (5). The effective action of the dispersive interaction is given by [30,31]: $(e^{i\varphi(\tau)\hat{n}}|e\rangle\langle e| + |g\rangle\langle g|)|\Psi_{(0)}\rangle_{AF}$, where $|\Psi_{(0)}\rangle_{AF} = |\Psi_{(0)}\rangle_A \otimes |\Psi_{(0)}\rangle_F$, with the initial atomic state $|\Psi_{(0)}\rangle_A = 1/\sqrt{2}(|g\rangle + |e\rangle)$ and the initial field state $|\Psi_{(0)}\rangle_F$ given by Eq. (5); $\varphi(\tau) = \Omega^2\tau/\delta$, Ω being the Rabi frequency, τ the duration of atom-field interation inside the cavity and $\delta = \omega - \omega_0$ is the detuning between atom and field frequencies. So, we easily find that the field-state in (5) rotates to the final state

$$|\Psi(\tau)\rangle = \frac{1}{\mathcal{N}'} \sum_{k=0}^{N} \mathrm{e}^{ik\varphi(\tau)} |k\rangle, \tag{10}$$

which coincides with the desired (truncated) phase state in Eq. (2) for $\mathcal{N}' = (1+N)^{-1/2}$ and $\varphi(\tau) = \theta_m$, where [1–3]: $\theta_m = \theta_0 + 2\pi m(1+N)^{-1}$.

In principle, an *exact* phase state is obtained in the limit $N \to \infty$ [1–3]. Here, a first impediment to get this goal comes see from decoherence effect (due to unavoidable interaction between the cavity-field with its environment). Decoherence means that our field state being prepared degrades in time, thus loosing its ‘integrity’ [28,29]. Decoherence effect enhances when the field excitation increases, hence it blows up in the limit $N \to \infty$.

Notwithstanding, the present scheme, besides clarifying the practical restrictions to the generation of an *exact* phase state for stationary fields, allows one to obtain approximate phase states, the so called ‘truncated phase states’, having low average excitation for small values of N.

For completeness, we mention an interesting point concerning to states having the ‘format’ of the phase state in Eq. (2). Firstly, we write it in the form

$$|\theta_m\rangle = \sum_{k=0}^{N} C_k \mathrm{e}^{ik\theta_m} |k\rangle, \tag{11}$$

with $C_k = 1/(1+N)$. Next, if we replace the coefficients $C_k = 1/(1+N)$ by $C_k = \mathrm{e}^{-|\alpha|^2/2} \alpha^k / \sqrt{k!}$ and assume that N is sufficiently large, we recognize in (11) the rotated coherent state $|\alpha'\rangle = |\alpha \mathrm{e}^{i\theta_m}\rangle$, whose generation is well studied in the literature [32]. In addition, if we also replace the phase θ_m in Eq. (11) by $\theta_m = \theta_m(k) = \lambda k^p$, with $p = 1,2,3,\ldots$ then the resulting state in (11) will no longer correspond to some coherent state. Instead, one would obtain another interesting result: a noncoherent state having Poissonian statistics (NCS) [33], whose generation was also studied in the literature [34]. So, in this context, the phase state, the rotated coherent state and the NCS can be viewed as alternative realizations of the state in (11).

3. Generation of phase state for *travelling* fields

Usually, the generation of a nonclassical field state for travelling waves is obtained by sending a light field conveniently chosen in an initial state $|\Psi(0)\rangle$ (e.g., a coherent state) through a nonlinear medium. The interaction of the field with this medium creates an emergent field in a desired state, via *unitary* evolution: $|\Psi(\tau)\rangle = \hat{U}(\tau)|\Psi(0)\rangle$, where τ is the atom-field interaction time and $\hat{U}(\tau)$ is the unitary evolution operator. Illustrative states of such ‘engineering’ are: the squeezed state [14–16] and the Schrödinger’s cat state [17], both concerning with states of light in travelling waves.

In the present case, we will follow a scheme recently introduced by Pegg et al. [18], using a pair of beam splitters in which the two output fields in the second beam-splitter are detected selectively. So the field created in one output of the first beam-splitter, now evolves nonunitarily to a desired state.

Fig. 2 is a sketch of the experimental arrangement for generation of phase state for a travelling field. In this figure BS-1 and BS-2 are two beam-splitters having transmittance (reflectance) t_i (r_i), $i = 1,2$; D_1 and D_2 are photodetectors; $|1\rangle_a$ and $|N-1\rangle_b$ are input states in the first beam-splitter; $|\alpha\rangle_c$ is a coherent state entering the second beam splitter; $|\phi\rangle_a$ and $|\phi\rangle_b$ are the outputs for BS-1, $|\phi\rangle_b$ being also one input in BS-2. Here, we will show that, by choosing the detector D_1 counting 1 photon while

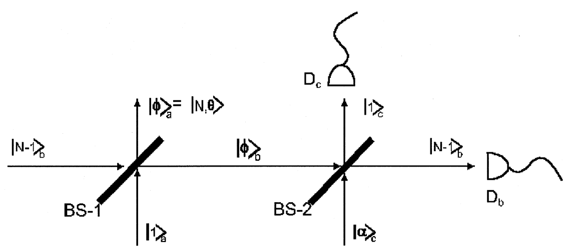

Fig. 2. The same as in Fig. 1, for *travelling* waves. BS-1 and BS-2 indicate the required beam splitters.

detecting $N-1$ photons in D_2, we will obtain the wanted state $|\phi\rangle_a = |N,\theta\rangle$, where $|N,\theta\rangle$ is the PB phase state given in Eq. (2). To this end, we have, step by step:

(i) *Generation of the state* $|1,\theta\rangle$

According to Fig. 2, for $N=1$ the input states in the BS-1 are $|1\rangle_a$ and $|0\rangle_b$. Hence, we can wright the *entire* input state entering the BS-1 as

$$|\Psi_{\text{in}}\rangle_{ab} = |1\rangle_a|0\rangle_b = \hat{a}^\dagger_{\text{in}}|0\rangle_a|0\rangle_b, \tag{12}$$

whereas the *entire* output in the BS-1 is given by

$$|\Psi_{\text{out}}\rangle_{ab} = \left(t_1\hat{a}^\dagger_{\text{out}} + ir_1\hat{b}^\dagger_{\text{out}}\right)|0\rangle_a|0\rangle_b, \tag{13}$$

where the beam-splitter equation (Walls et al. [31])

$$\begin{pmatrix}\hat{a}^\dagger_{\text{in}}\\ \hat{b}^\dagger_{\text{in}}\end{pmatrix} = \begin{pmatrix} t_1 & ir_1\\ ir_1 & t_1\end{pmatrix}\begin{pmatrix}\hat{a}^\dagger_{\text{out}}\\ \hat{b}^\dagger_{\text{out}}\end{pmatrix} \tag{14}$$

has been used. Next, the application of the Eqs. (13), (14) gives the output state

$$|\Psi_{\text{out}}\rangle_{ab} = i\eta_1 t_1\left\{|0\rangle_a|1\rangle_b - \frac{i}{\eta_1}|1\rangle_a|0\rangle_b\right\}, \tag{15}$$

where $\eta_i = r_i/t_i,\ i=1,2$.

Now, it follows $|\phi\rangle_a = |1,\theta\rangle$, when the state $|\phi\rangle_b$ has the form

$$|\phi\rangle_b = \frac{i}{\sqrt{2}\,\eta_1 t_1}\left\{|1\rangle_b - i\eta_1\mathrm{e}^{-i\theta}|0\rangle_b\right\}, \tag{16}$$

since $|\phi\rangle_a$ is obtained from the projection

$$|\phi\rangle_a = {}_b\langle\phi|\Psi_{\text{out}}\rangle_{ab}. \tag{17}$$

On the other hand, we have for the BS-2 the output state

$$|\Psi_{\text{out}}\rangle_{bc} = |0\rangle_b|1\rangle_c = \hat{c}^\dagger_{\text{out}}|0\rangle_b|0\rangle_c, \tag{18}$$

so, the entire input in the BS-2 is given by

$$|\Psi_{\text{in}}\rangle_{bc} = \left(t_2\hat{c}^\dagger_{\text{in}} - ir_2\hat{b}^\dagger_{\text{in}}\right)|0\rangle_a|0\rangle_b, \tag{19}$$

where we have employed the beam-splitter equation

$$\begin{pmatrix}\hat{b}^\dagger_{\text{out}}\\ \hat{c}^\dagger_{\text{out}}\end{pmatrix} = \begin{pmatrix} t_1 & -ir_1\\ -ir_1 & t_1\end{pmatrix}\begin{pmatrix}\hat{b}^\dagger_{\text{in}}\\ \hat{c}^\dagger_{\text{in}}\end{pmatrix}. \tag{20}$$

Thus, the straightforward application of the Eq. (20) in the Eq. (19) results

$$|\Psi_{\text{in}}\rangle_{bc} = -i\eta_2 t_2\left\{|1\rangle_b|0\rangle_c + \frac{i}{\eta_2}|0\rangle_b|1\rangle_c\right\}, \tag{21}$$

and application of the equation ($\mathscr{K}$ stands for normalization)

$$|\phi\rangle_b = \mathscr{K}_c\langle\alpha|\Psi_{\text{in}}\rangle_{bc}, \tag{22}$$

leads us to get

$$|\phi\rangle_b = -i\eta_2 t_2\mathscr{K}\mathrm{e}^{-R^2/2}\left\{|1\rangle_b + \frac{iR\mathrm{e}^{-i\theta}}{\eta_2}|0\rangle_b\right\}, \tag{23}$$

where $|\alpha\rangle$ is a coherent state (with $\alpha = R\mathrm{e}^{i\theta}$)

$$|\alpha\rangle = \mathrm{e}^{-R^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{24}$$

At this point, comparing the Eqs. (16) and (23) we find that

$$\prod_{i=1}^{2}\eta_i = -R, \tag{25}$$

which gives the relationship among the unknowns parameters η_1, η_2 and R, yielding the state $|1,\theta\rangle$. One solution, e.g., is given by the set: $\{\eta_1,\eta_2,R\} = \{10,\ 0.5,\ -5\}$.

(ii) *Generation of the state* $|2,\theta\rangle$

In this case, instead of starting from the input state in the BS-1: $|1\rangle_a\ |0\rangle_b$, as before, we will start from the input state $|\Psi_{\text{in}}\rangle_{ab} = |1\rangle_a|1\rangle_b = \hat{a}^\dagger_{\text{in}}\hat{b}^\dagger_{\text{in}}|0\rangle_a|0\rangle_b$, and following the previous procedure we obtain the output state in the b-arm of the BS-1

$$|\phi\rangle_b = {}_a\langle\phi|\Psi_{\text{out}}\rangle_{ab} = \frac{i}{\sqrt{6}\,\eta_1 t_1^2}\times\left\{|2\rangle_b + \frac{i\sqrt{2}\,\eta_1}{(\eta_1^2-1)}\mathrm{e}^{-i\theta}|1\rangle_b + \mathrm{e}^{-2i\theta}|0\rangle_b\right\}. \tag{26}$$

The same mimic applied to the BS-2, with $|\Psi_{\rm out}\rangle_{bc} = |1\rangle_b|1\rangle_c = \hat{b}^\dagger_{\rm out}\hat{c}^\dagger_{\rm out}|0\rangle_b|0\rangle_c$, gives the input state in the b-arm of BS-2

$$|\phi\rangle_b = \mathscr{K}_c\langle\alpha|\Psi_{\rm in}\rangle_{bc} = -i\sqrt{2}\,\eta_2 t_2^2 \mathscr{K} e^{-R^2/2} \times\left\{|2\rangle_b - i\frac{R(\eta_2^2-1)e^{-i\theta}}{\sqrt{2}\,\eta_2}|1\rangle_b + \frac{R^2}{\sqrt{2}}e^{-2i\theta}|0\rangle_b\right\}, \tag{27}$$

whose comparison with (26) gives the equations connecting η_1, η_2 and R

$$\prod_{i=1}^{2}\frac{\eta_i}{(\eta_i^2-1)} = -\frac{R}{2}, \tag{28a}$$

$$\prod_{i=1}^{2}\frac{\eta_i}{\eta_i} = \frac{R^2}{\sqrt{2}}, \tag{28b}$$

one solution being given by the set: $\{\eta_1,\eta_2,R\} = \{0.9,\ 8.1,\ 1.19\}$.

(iii) *Generation of the state* $|3,\theta\rangle$

Here, we must start, in the BS-1. with the input $|\phi_{\rm in}\rangle_{ab} = |1\rangle_a|2\rangle_b$. The application of the foregoing procedure furnishes, for this case,

$$|\phi\rangle_b = {}_a\langle\phi|\Psi_{\rm out}\rangle_{ab} = \frac{i}{2\sqrt{6}\,\eta_1 t_1^3}\left\{|3\rangle_b + i\frac{\sqrt{3}\,\eta_1}{2\eta_1^2-1}e^{-i\theta}|2\rangle_b - \frac{\sqrt{3}}{\eta_1^2-2}e^{-2i\theta}|1\rangle_b + i\frac{1}{\eta_1}e^{-3i\theta}|0\rangle_b\right\}, \tag{29}$$

and

$$|\phi\rangle_b = \mathscr{K}_c\langle\alpha|\Psi_{\rm in}\rangle_{bc} = -i\sqrt{3}\,\eta_2 t_2^3 \mathscr{K} e^{-R^2/2} \times\left\{|3\rangle_b - i\frac{R(2\eta_2^2-1)}{\sqrt{3}\,\eta_2}e^{-i\theta}|2\rangle_b - \frac{R^2(\eta_2^2-2)}{\sqrt{6}}e^{-2i\theta}|1\rangle_b - i\frac{R^3\eta_2}{\sqrt{6}}e^{-3i\theta}|0\rangle_b\right\}, \tag{30}$$

whose comparison leads to the relations for to the state $|3,\theta\rangle$

$$\prod_{i=1}^{2}\frac{\eta_i}{2\eta_i^2-1} = -\frac{R}{3}, \tag{31a}$$

$$\prod_{i=1}^{2}\frac{1}{\eta_i^2-2} = \frac{R^2}{3\sqrt{2}}, \tag{31b}$$

$$\prod_{i=1}^{2}\frac{1}{\eta_i} = -\frac{R^3}{\sqrt{6}}, \tag{31c}$$

with solution $\{\eta_1,\eta_2,R\} = \{0.44337,\ 1.0955,\ 1.7149\}$.

(iv) *Generation of the state* $|4,\theta\rangle$

In this case, a lengthy but straightforward calculation, starting from the input stat $|1\rangle_a|3\rangle_b$ gives

$$|\phi\rangle_b = {}_a\langle\phi|\Psi_{\rm out}\rangle_{ab} = \frac{i}{\sqrt{5!}\,\eta_1 t_1^4}\times\left\{|4\rangle_b + i\frac{2\eta_1}{3\eta_1^2-1}e^{-i\theta}|3\rangle_b - \frac{\sqrt{6}}{3(\eta_1^2-1)}e^{-2i\theta}|2\rangle_b + -i\frac{2}{\eta_1^3-3\eta_1}e^{-3i\theta}|1\rangle_b - \frac{1}{\eta_1^2}e^{-4i\theta}|0\rangle_b\right\}, \tag{32}$$

and

$$|\phi\rangle_b = \mathscr{K}_c\langle\alpha|\Psi_{\rm in}\rangle_{bc} = -i2\eta_2 t_2^4 \mathscr{K} e^{-R^2/2}\left\{|4\rangle_b - i\frac{R(3\eta_2^2-1)}{2\eta_2}\times e^{-i\theta}|3\rangle_b - \frac{3R^2(\eta_2^2-1)}{2\sqrt{3}}e^{-2i\theta}|2\rangle_b + {}+ i\frac{R^3(\eta_2^3-3\eta_2)}{2\sqrt{6}}e^{-3i\theta}|1\rangle_b - \frac{R^4\eta_2^2}{2\sqrt{6}}e^{-4i\theta}|0\rangle_b\right\}. \tag{33}$$

Comparison of Eqs. (32), (33) allows one to get

$$\prod_{i=1}^{2}\frac{\eta_i}{3\eta_i^2-1} = -\frac{R}{4}, \tag{34a}$$

$$\prod_{i=1}^{2} \frac{1}{9(\eta_i^2-1)} = \frac{R^2}{6\sqrt{2}}, \tag{34b}$$

$$\prod_{i=1}^{2} \frac{1}{\eta_i^3-3\eta_i} = -\frac{R^3}{4\sqrt{6}}, \tag{34c}$$

$$\prod_{i=1}^{2} \frac{1}{\eta_i^2} = \frac{R^4}{2\sqrt{6}}, \tag{34d}$$

with the solution $\{\eta_1, \eta_2, R\} = \{40, 9, 0.01\}$.

Here, it is worth noting that, for the state $|4,\theta\rangle$ we have found 4 equations [see Eqs. (34)] and 3 unknowns (η_1, η_2, R). The system will have solution only if one of the equations is a linear combination of the remaining ones. We have verified that this happens. So, we may try to find a recurrence formula connecting the variables η_1, η_2, and R for the general case $|N,\theta\rangle$. A little inspection on the experimental scheme [Fig. 2] shows that: for our choice on selective detection the entire input in the BS-1 coincides with the entire output in the BS-2. So the net result of our experimental arrangement is the transformation of the coherent state $|\alpha\rangle_c$ entering the BS-2 into the wanted state $|\phi\rangle_a = |N,\theta\rangle$ emerging in the BS-1. Now, the ingredient directly responsible for this transformation is the state $|\phi\rangle_b$, which connects the states $|\alpha\rangle_c$ and $|N,\theta\rangle$. Next, a careful inspection on the state $|\phi\rangle_b$, step by step as constructed before, allows us to get its generalized expression when emerging from the BS-1

$$\begin{aligned}|\phi\rangle_b &= {}_a\langle\phi|\Psi_{\text{out}}\rangle_{ab} \\ &= \frac{i}{\eta_1 t_1^N \sqrt{(N+1)!}} \sum_{n=0}^{N} \frac{(i)^n \sqrt{N!}}{\sqrt{n!(N-n)!}} \\ &\quad \times \frac{\eta_1}{A_{N,n}\eta_1^{n+1} - B_{N,n}\eta_1^{n-1}} e^{-in\theta}|N-n\rangle,\end{aligned} \tag{35}$$

and its generalized expression when reaching the BS-2

$$\begin{aligned}|\phi\rangle_b &= \mathcal{K}_c\langle\alpha|\Psi_{\text{in}}\rangle_{bc} \\ &= -\frac{i\eta_2 t_2^N \sqrt{N!}\,\mathcal{K} e^{-R^2/2}}{\sqrt{(N-1)!}} \sum_{n=0}^{N} (-i)^n R^n \\ &\quad \times \frac{\sqrt{(N-n)!}}{\sqrt{N!}} \frac{A_{N,n}\eta_2^{n+1} - B_{N,n}\eta_2^{n-1}}{\eta_2} \\ &\quad \times e^{-in\theta}|N-n\rangle.\end{aligned} \tag{36}$$

The identification of (35) with (36) gives

$$\begin{aligned}&\prod_{i=1}^{2} \frac{\eta_i}{A_{N,n}\eta_i^{n+1} - B_{N,n}\eta_i^{n-1}} \\ &\quad = \frac{(-1)^n (N-n)! \sqrt{n!}\, R^n}{N!},\end{aligned} \tag{37}$$

where

$$A_{N,n} = \frac{N-n}{N}\binom{N}{n}, \tag{38a}$$

$$B_{N,n} = \frac{n}{N}\binom{N}{n}. \tag{38b}$$

Since $n = 0, 1, 2, \dots, N$, then the Eq. (37) implies in N equations, one of the $N+1$ equations being redundant.

4. Comments and conclusion

We have employed some existing schemes of QSE to present two proposals for the generation of a phase state: one of them concerning with the generation of a phase state for a *stationary* field; the other concerns with the same for a *travelling* field. Since an exact phase state is obtained in the limit $N \to \infty$ [1–3] then it involves infinite energy [5] ($\bar{n} = N/2 \to \infty$ when $N \to \infty$) then it is interesting to show the way in which such natural restriction manifests in these experiments. For stationary fields it appears connected with the efficiency of atomic detection involved in the experiment, which is proportional to $1/2^N$ and goes to zero in the required limit $N \to \infty$. For travelling fields the same restriction appears, now connected with the efficience of detection in the outputs $(|1\rangle_a, |N-1\rangle_c)$ of the BS-2, which also tends to zero in the limit $N \to \infty$ since there are $2N$ different outcomes given by: $\{|N-j\rangle_c, |j\rangle_b\}$ and $\{|j\rangle_c, |N-j\rangle_b\}$, with $j = 0, 1, 2, \dots N-1$. In spite of the natural restriction to the generation of an exact phase-state, we have shown how to construct approximations to this state, the so called ‘truncated phase states’, for both cases of *stationary* and *travelling* fields.

Let us account for some sensitive points in the above-presented scheme for QSE. These points also concerns to Refs. [18,19] and [35]. First, we note that the efficiency for single-photon detectors is about 70%, while the damping constant for BS's is considerably small, less than 2% in BK7 crystals. In addition to the non-unity quantum efficiency detectors and the absorptive beam splitters, there are two other experimental non-idealities that seem to be equally important. The first is the present technological difficulty to dispose of arbitrary number states, necessary in the experiment: for example, we do not have perfect single photon source, as required in the a-arm of the BS-1. The commonly cited method of parametric fluorescence only approximates a single photon source, and this approximation must be evaluated in its effect on projection synthesis technique. The same problem also concerns with number state $|N-1\rangle$, required in the b-arm of BS-1. Here we mention the recent quantum Fock filter scheme proposed by D'Ariano et. at. [36]. Such scheme is able to select a specific Fock component and superpositions of few number states from a generic input state. So, the device proposed in Ref. [36] plays a crucial role in the present QSE scheme. The second difficulty to be circumvented concerns with detectors required to discriminate between $0,1,2,\ldots N-1$ photons arrivals. Since this is not available presently, an accurate analysis of the errors should also account for this deficiency.

As a final remark, we should stress that for the choice $\theta_m = 0$ our state will correspond to a generalization of a previous result by Pegg, Phillips and Barnett [18]. In this special case, following the nomenclature of Ref. [18] the present device should be named generalized 'quantum scissors'.

Acknowledgements

This paper was partially supported by CAPES (Y.G.), FAPESP (C.J.V.B.) and CNPq (B.B., M.H.Y.M.), Brazilian agencies.

References

[1] D.T. Pegg, S.M. Barnett, Europhys. Lett. 6 (1988) 483.
[2] D.T. Pegg, S.M. Barnett, J. Mod. Opt. 36 (1988) 7.
[3] D.T. Pegg, S.M. Barnett, Phys. Rev. A 39 (1989) 1665.
[4] M. Hillery, Phys. Lett. A 111 (1985) 409.
[5] R. Loudon, The Quantum Theory of Light, Clarendon, Oxford, 1973.
[6] P.A.M. Dirac, Proc. R. Soc. (London) Ser. A 114 (1927) 243.
[7] F. London, Z. Phys. 40 (1927) 193.
[8] H.W. Louisell, Phys. Lett. 7 (1963) 60.
[9] L. Susskind, J. Glogower, Physics 1 (1964) 49.
[10] P. Carruthers, M.M. Nieto, Rev. Mod. Phys. 40 (1968) 411.
[11] J.W. Noh, A. Fougères, L. Mandel, Phys. Rev. Lett. 71 (1993) 2579.
[12] J.R. Torgerson, L. Mandel, Phys. Rev. Lett. 76 (1996) 3939.
[13] Yu.I. Vorontsov, Yu.A. Rembovsky, Phys. Lett. A 254 (1999) 7.
[14] R.L. Robinson, Science 230 (1985) 927.
[15] R.L. Robinson, Science 233 (1986) 280.
[16] R.E. Slusher, L.W. Holber, B. Yurke, J.C. Mertz, F.W. Walley, Phys. Rev. Lett. 55 (1985) 2409.
[17] B. Yurke, D. Stoler, Phys. Rev. Lett. 57 (1986) 13.
[18] D.T. Pegg, L.S. Phillips, S.M. Barnett, Phys. Rev. Lett. 81 (1998) 1604.
[19] M. Dakna, J. Clausen, L. Knöll, D.-G. Welsch, Phys. Rev. A 59 (1999) 1658.
[20] S. Haroche, in: J. Dalibard, J.M. Raimond, J. Zin-Justin (Eds.), Fundamental Systems in Quantum Optics, Les Houches, 1990, Elsevier, NY, 1992, p. 771.
[21] S. Zheng, G. Guo, Opt. Commun. 133 (1997) 139.
[22] S. Haroche, M. Brune, J.M. Raimond, Appl. Phys. B 54 (1992) 355.
[23] M. Brune, L. Davidovich, S. Haroche, J.M. Raimond, N. Zagury, Phys. Rev. A 45 (1992) 5193.
[24] M.H.Y. Moussa, B. Baseia, Phys. Lett. A 238 (1998) 223.
[25] K. Vogel, V.M. Akulin, W.P. Schleich, Phys. Rev. Lett. 71 (1993) 1816.
[26] M. Brune, P. Nussenzveig, F. Schmidt-Kaler, F. Bernadot, A. Maali, J.M. Raimond, S. Haroche, Phys. Rev. Lett. 72 (1994) 3339.
[27] L. Davidovich, N. Zagury, M. Brune, J.M. Raimond, S. Haroche, Phys. Rev. A 50 (1994) 895.
[28] M. Brune, S. Haroche, J.M. Raimond, L. Davidovich, N. Zagury, Phys. Have. A 45 (1992) 5193.
[29] L. Davidovich, N. Zagure, M. Brune, J.M. Raimond, S. Haroche, Phys. Rev. A 50 (1994) R895.
[30] T. Sleator, H. Weinfurter, Phys. Rev. Lett. 74 (1995) 4087.
[31] D.F. Walls, G.J. Milburn, Quantum Optics, Springer, Berlin, 1994.
[32] L. Davidovich, A. Maali, M. Brune, J.M. Raimond, S. Haroche, Phys. Rev. Lett. 71 (1993) 2360.
[33] C. Dantas, B. Baseia, Physica A 265 (1999) 176.
[34] M. Kitagawa, Y. Yamamoto, Phys. Rev. A 34 (1986) 3974, see also Ref. [17].
[35] D.T. Pegg, S.M. Barnett, Phys. Rev. Lett. 76 (1996) 4148.
[36] G.M. D'Ariano, L. Maccone, M.G.A. Paris, M.F. Sacchi, quantph/9904074.