

Particle Creation in a Robertson–Walker Universe Revisited

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Abstract We reanalyze the problem of particle creation in a $3 + 1$ spatially closed Robertson–Walker space-time. We compute the total number of particles produced by this non-stationary gravitational background as well as the corresponding total energy and find a slight discrepancy between our results and those recently obtained in the literature (Setare, M.R. in Int. J. Theor. Phys. 43:2237, 2004).

1 Introduction

In a curved space-time the concept of particles is a subtle one and the meaning of what is a particle or what is a detector becomes much more difficult than in a flat space-time [3]. Basically, this happens because in a general curved space-time we do not have Lorentz symmetry anymore and it is precisely this symmetry that, in a flat space-time, allows us to identify the best vacuum state for the theory. Lorentz symmetry assures that the vacuum state is the same for all inertial observers.

Fortunately, in many problems of our interest, there are regions where a choice of the physical vacuum is quite natural. That happens when the time dependence of the field in these regions is harmonic (or, at least, almost harmonic). A basic condition to have a field with a harmonic time dependence is that the background metric and the eventual boundaries do not depend on time. In some specific cases, the background metric respects this condition only in asymptotic times (distant future and remote past). In these cases, the physical interpretation of particles is possible at those asymptotical times, but not between them. As a consequence, a non-static curved space-time background may lead to the phenomenon of particle creation.

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The first one to discuss the problem of particle creation due to a curved space-time background was Schrödinger in 1939 [11], but the first one that carefully investigated this phenomenon was Parker in the late 60’s [8–10]. In 1974, Hawking [6] observed that black holes can produce particles as a consequence of quantum effects (see [4] and references therein for recent works in Hawking radiation). The particle creation in a 1 + 1 spatially closed Robertson–Walker space-time was investigated in [2, 3]. The 3 + 1 version of this problem has been investigated in a recent work [12].¹ However, when generalizing the 1 + 1 solution to the 3 + 1 one, this author missed a degeneracy factor leading to an incomplete answer for the total number of particles created as well as the corresponding total energy.

The main purpose of this short paper is to compute the total number of particles created and the total energy related to them taking into account the correct degeneracy factor that arises when we are in a 3 + 1 dimensions. With the aid of appropriated graphics, we also make a brief discussion on how the parameters that appear in the metric affect the total number of particles produced.

2 Particle Creation

We consider the case of an expanding spatially closed Robertson–Walker universe whose line element and scalar curvature are given, respectively, by

$$\begin{aligned}
 ds^2 &= a^2(\eta)(d\eta^2 - d\rho^2 - \text{sen}^2\rho(d\theta^2 + \text{sen}^2\theta d\varphi)), \\
 R &= 6a^{-3}(\partial_\eta^2 a + a),
 \end{aligned}
 \tag{1}$$

where $a(\eta)$ is the scale factor, η is the conformal time and $0 \leq \rho \leq \pi$. In this case the equation of a massive scalar field conformally coupled to the metric is written as

$$\frac{\partial^2 u_{\mathbf{k}}}{\partial \eta^2} + \frac{2}{a} \frac{\partial a}{\partial \eta} \frac{\partial u_{\mathbf{k}}}{\partial \eta} + \left(m^2 a^2 + 1 + \frac{1}{a} \frac{\partial^2 a}{\partial \eta^2} \right) u_{\mathbf{k}} - \nabla_{\text{ang}} u_{\mathbf{k}} = 0,
 \tag{2}$$

where ∇_{ang} is the angular part of the Laplacian operator on a 3-sphere. This operator has the hyper-spherical harmonics as eigenfunctions, that satisfy the following differential equation [5]

$$\nabla_{\text{ang}} Y(l, m_1, m_2; \varphi, \theta, \rho) = -l(l + 2)Y(l, m_1, m_2; \varphi, \theta, \rho),
 \tag{3}$$

where $l = 0, 1, 2, \dots$, $m_1 = 0, \pm 1, \pm 2, \dots, \pm l$ and $m_2 = 0, \pm 1, \pm 2, \dots, \pm m_1$. Hence, the ∇_{ang} in (2) suggests the *ansatz*,

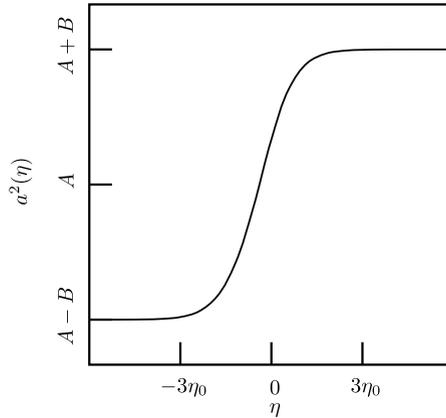
$$u_{\mathbf{k}}(\varphi, \theta, \rho, \eta) = \frac{1}{a(\eta)} Y(l, m_1, m_2; \varphi, \theta, \rho) g_l(\eta).
 \tag{4}$$

Substituting (4) in (2), we obtain the differential equation for $g_l(\eta)$,

$$\frac{d^2 g_l(\eta)}{d\eta^2} + \omega_l^2(\eta) g_l(\eta) = 0,
 \tag{5}$$

¹This problem also was investigated in [7], but the authors have not written a closed analytical expression for the total number of particles created.

Fig. 1 Graphic of $a^2(\eta)$ versus η for a closed expanding universe



where

$$\omega_l^2(\eta) = (l + 1)^2 + m^2 a^2(\eta). \tag{6}$$

Note that this equation is similar to that of a mechanical harmonic oscillator with a time-dependent frequency.

Now, we consider an exactly solvable case due to a convenient choice of $a(\eta)$, namely

$$a(\eta) = \sqrt{A + B \tanh\left(\frac{\eta}{\eta_0}\right)}, \tag{7}$$

where A, B and η_0 are constants and $A > B$. Since we are considering an expanding universe, $B > 0$ and $\eta_0 > 0$. An inspection in Fig. 1 allow us to interpret η_0 as the time scale of the expansion of the Universe while $A - B$ and $A + B$ are, respectively, the scales of the size of the universe before and after the expansion.

As we can see from Fig. 1, there are two asymptotic times where the background metric is almost static, namely, the remote pass, characterized by $\eta \ll -\eta_0$ and the distant future, characterized by $\eta \gg \eta_0$. At those asymptotic times the corresponding vacuum states may be well defined if the solutions of (5) have the following asymptotic limit

$$\lim_{\eta \rightarrow \mp \infty} g_l^{(p,f)}(\eta) = \frac{e^{-i\omega_l^{(p,f)}\eta}}{\sqrt{2\omega_l^{(p,f)} N_{\mathbf{k}}}}, \quad \omega_l^{(p,f)} = \sqrt{(l + 1)^2 + m^2(A \mp B)}, \tag{8}$$

where the superscripts p and f stand for past and future solutions, respectively. The solutions of (5) that respect the asymptotic limits (8) are given, respectively, by

$$g_l^p = \frac{\xi^{\frac{1}{2}\omega_l^f \eta_0} (1 - \xi)^{-\frac{1}{2}\omega_l^p \eta_0}}{(2\omega_l^p N_{\mathbf{k}})^{1/2}} {}_2F_1(1 + i\omega_l^- \eta_0, i\omega_l^- \eta_0; 1 - i\omega_l^p \eta_0; 1 - \xi), \tag{9}$$

$$g_l^f = \frac{\xi^{\frac{1}{2}\omega_l^f \eta_0} (1 - \xi)^{-\frac{1}{2}\omega_l^p \eta_0}}{(2\omega_l^f N_{\mathbf{k}})^{1/2}} {}_2F_1(1 + i\omega_l^- \eta_0, i\omega_l^- \eta_0; 1 + i\omega_l^f \eta_0; \xi),$$

where ${}_2F_1$ is the hypergeometric function [1] and we have defined $\omega_l^{(\pm)} := \frac{1}{2}(\omega_l^f \pm \omega_l^p)$ and $\xi := (1 - e^{2\eta/\eta_0})^{-1}$. Since $u_{\mathbf{k}}^p$ and $u_{\mathbf{k}}^f$ are not equal, the correspond-

ing Bogolubov coefficients must be non-vanishing. Using some well known proprieties of the hypergeometric function, we can write $u_{\mathbf{k}}^p$ in terms of $u_{\mathbf{k}}^f$ and $u_{\mathbf{k}}^{f*}$ as follows

$$u_{\mathbf{k}}^p = \sum_{\mathbf{k}'} (\alpha_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'}^f + \beta_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'}^{f*}), \tag{10}$$

with the Bogolubov coefficients given by

$$\begin{aligned} \alpha_{\mathbf{k}\mathbf{k}'} &= \delta_{\mathbf{k}\mathbf{k}'} \sqrt{\frac{\omega_l^f}{\omega_l^p}} \frac{\Gamma(1 - i\omega_l^p \eta_0) \Gamma(-i\omega_l^f \eta_0)}{\Gamma(-i\omega_l^+ \eta_0) \Gamma(1 - i\omega_l^+ \eta_0)}, \\ \beta_{\mathbf{k}\mathbf{k}'} &= \delta_{\mathbf{k}\mathbf{k}'} \sqrt{\frac{\omega_l^f}{\omega_l^p}} \frac{\Gamma(1 - i\omega_l^p \eta_0) \Gamma(i\omega_l^f \eta_0)}{\Gamma(i\omega_l^- \eta_0) \Gamma(1 + i\omega_l^- \eta_0)}. \end{aligned} \tag{11}$$

In the remote past all the inertial particle detectors register the complete absence of particles in the state $|0^p\rangle$ (the vacuum state associated to $u_{\mathbf{k}}^p$). However, in the distant future, any inertial particle detector will register a number of particles with quantum numbers \mathbf{k} in the $|0^p\rangle$ state given by

$$\langle 0^p | N_{\mathbf{k}}^f | 0^p \rangle = \sum_{\mathbf{k}'} |\beta_{\mathbf{k}\mathbf{k}'}|^2 = \frac{\sinh^2(\pi \omega_l^{(-)} \eta_0)}{\sinh(\pi \omega_l^p \eta_0) \sinh(\pi \omega_l^f \eta_0)}. \tag{12}$$

The total number of produced particles and the total energy associated to them are given, respectively, by

$$\langle 0^p | N^f | 0^p \rangle = \sum_{\mathbf{k}'} \sum_{\mathbf{k}} |\beta_{\mathbf{k}\mathbf{k}'}|^2 = \sum_{l=0}^{\infty} (l+1)^2 \frac{\sinh^2(\pi \omega_l^{(-)} \eta_0)}{\sinh(\pi \omega_l^p \eta_0) \sinh(\pi \omega_l^f \eta_0)}, \tag{13}$$

$$E = \sum_{\mathbf{k}} \langle 0^p | N_{\mathbf{k}}^f | 0^p \rangle \omega_l^f = \sum_{l=0}^{\infty} (l+1)^2 \frac{\omega_l^f \sinh^2(\pi \omega_l^{(-)} \eta_0)}{\sinh(\pi \omega_l^p \eta_0) \sinh(\pi \omega_l^f \eta_0)}. \tag{14}$$

Since we were not able to perform summation (13) analytically, let us make a numerical analysis of the preceding results constructing, for instance, the graphic of the total number of particles created N versus the mass of the field excitations m . Naively, we could expect that the total number of particles created was a monotonically decreasing function of m . However, as we can see from Fig. 2, there is a value m_0 for the mass of the field at which the total number of created particles $N(m_0)$ reaches a maximum.

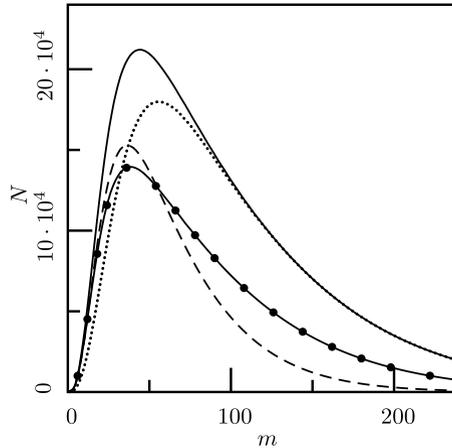
It is convenient to define a width Δm_0 as it is done in a Gaussian distribution. With this purpose, we define m_+ and m_- such that

$$N(m_0 + m_+) = N(m_0 - m_-) = e^{-1} N(m_0), \tag{15}$$

so that $\Delta m_0 := m_+ - m_-$.

Note from (13) that $N(m_0)$, m_0 and Δm_0 depend only on $A - B$, $A + B$ and η_0 . Despite the fact that we were not able to get a closed analytic expression for m_0 , Δm_0 and $N(m_0)$ as a function of the metric parameters $A - B$, $A + B$ and η_0 , we can see numerically how these quantities depend on those parameters. We do that by plotting the graph of the total number of created particles as a function of the field mass for several values of $A - B$, $A + B$ and η_0 (Fig. 2). At first we plot a control curve (in Fig. 2, it is represented as a continuous line),

Fig. 2 Graphic of the total number of created particles N versus the field mass m for different values of parameters $A - B$, $A + B$ and η_0 . The *continuous line* represents the case where $\eta_0 = 0.01$, $A - B = 1$ and $A + B = 101$. The *dashed line* represents the case where $\eta_0 = 0.01$, $A - B = 3.25$ and $A + B = 101$. The *dotted line* represents the case where $\eta_0 = 0.01$, $A - B = 1$ and $A + B = 51$. Finally, the *continuous dotted line* represents the case where $\eta_0 = 0.0115$, $A - B = 1$ and $A + B = 101$



and then we modify one of the parameters, plot a new curve and compare to the control one. We repeat this procedure for all parameters.

Changing the parameters $A - B$, $A + B$ and η_0 as we did in Fig. 2 and then analyzing numerically what happens to the graphic of N versus m , suggests the following behaviors: $N(m_0)$ gets larger and Δm_0 gets smaller as $A - B$ and η_0 decrease or $A + B$ increases. On the other hand m_0 gets larger as $A - B$, $A + B$ or η_0 increase.

3 Conclusions

In this work we computed the total number of particles created and the total energy related to them in a $3 + 1$ spatially closed Robertson–Walker space-time. By using appropriated graphics, we also discussed how the parameters that appear in the metric affect the total number of particles produced. We found an unexpected behavior of N as a function of m , namely, starting from zero, it increases until it reaches a maximum value at m_0 and after that it decreases monotonically as m increases.

The essential difference between our calculations and that presented in [12] is that this author does not take into account the degeneracy factor $(l + 1)^2$ that appears in (13) and (14). This factor must be included due to the degeneracy in the quantum numbers m_1 and m_2 . We think our discussion may be of some help in the analysis of similar problems, as for example, if one considers other functions $a(\eta)$ with behaviors slightly different from that one considered here.

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