Recurrence formula for generalized optical state truncation by projection synthesis

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A recent work [Pegg *et al.*, Phys. Rev. Lett. **81**, 1604 (1998)] showed how to generate a running-wave superposition of zero- and one-photon field states, $C_0|0\rangle + C_1|1\rangle$, by physical truncation of the photon number superposition making up a coherent state. Here we discuss the extension of this device to the case of N components: $C_0|0\rangle + C_1|1\rangle + \cdots + C_N|N\rangle$.

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Several interesting states of the quantized electromagnetic field are studied nowadays. Among them one can cite the number and (its complementary) phase states [1], the coherent [2] and squeezed states [3], superposition states having the previous basic states as components [4], the interpolating states [5], which vary between two limiting states, going from one to the other, etc. So, in view of the existence of an abundance of states in quantum optics, it is of interest to know how to prepare these states experimentally, this procedure being known in the literature as ''quantum state engineering'' [6]. The engineering schemes may consider either the case of stationary waves prepared inside a (high-Q) cavity [6,7] or the case of traveling waves [8,9]. In the realm of cavity QED phenomena, Vogel *et al.* [6] employed a resonant atom-field interaction to build up a trapped field in an initially empty cavity, while the proposal in Ref. [10] considers both resonant and dispersive atom-field interactions for the preparation of a general superposition in the empty cavity. An alternative proposal has been presented [11], named the *sculpture* of quantum states, where a coherent state is previously injected into the cavity and the Wigner distribution function of the desired state is sculptured, through atom-field interaction, from that representing the initial coherent state, the atoms playing the role of quantum chisels.

In recent work by Pegg, Phillips, and Barnett [8], preparation of an arbitrary running-wave superposition of the vacuum and one-photon states, $C_0|0\rangle + C_1|1\rangle$, without using cavities was demonstrated. In this way, a traveling field would be available for further applications, including performing measurements on other field states [12,13]. The scheme in [8] obtains the above mentioned superposition by physical truncation of the photon number superposition making up a coherent state. The proposal requires no additional extension of current experiments and is reasonably insensitive to photodetection efficiency for the fields most likely to be used in practice. Since the scheme works via a truncation of the Hilbert space, it has been called a ''quantum scissors'' device [8]. As mentioned in [8], states including superpositions of higher photon numbers might be fabricated by superposing fields prepared as superpositions of zero- and one-photon number states [14]. In this connection, we will present here an alternative way to prepare arbitrary truncated states, i.e., $C_0|0\rangle + C_1|1\rangle + C_2|2\rangle + \cdots + C_N|N\rangle$, $N = 1,2,3,\ldots$. The *method* is a direct extension of that in [8], called the optical truncation of a state by projection synthesis.

Figure 1 shows a sketch of the experimental setup proposed in [8] to generate a truncated state $C_0|0\rangle + C_1|1\rangle$ ($N = 1$). In this figure a single-photon field incident on a 50-50 symmetric beam splitter 1 (BS1), i.e., the input state $|\Psi_{in}\rangle_{ab} = |1\rangle_a|0\rangle_b$, generates a field state

$$|\Psi\rangle_{ab} = \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b + i|0\rangle_a|1\rangle_b), \tag{1}$$

with the subscripts standing for the two output modes a and b. The action of BS1 is described by the unitary operator [15]

$$\hat{R}_{ab} = \exp[i\theta_1(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)], \tag{2}$$

where $t_1 = \cos\theta_1$ and $r_1 = \sin\theta_1$ are the BS1 reflection and transmission coefficients, respectively. For the case of a 50-50 symmetric BS1 in [8], it follows that $\theta_1 = \pi/4$ and $t_1 = r_1 = 1/\sqrt{2}$.

By measuring the field mode b in the state $|\phi\rangle_b = C_1^*|1\rangle_b + iC_0^*|0\rangle_b$, we synthesize the projection of the field mode a in the desired superposition $|\phi\rangle_a = {}_b\langle\phi|\Psi\rangle_{ab} = C_0|0\rangle_a + C_1|1\rangle_a$. In [8] this procedure was accomplished by superposing on the field mode b an auxiliary coherent field in mode c, $|\alpha\rangle_c = \Sigma_{n=0}^{\infty}\alpha_n|n\rangle$, at a second 50-50 sym-

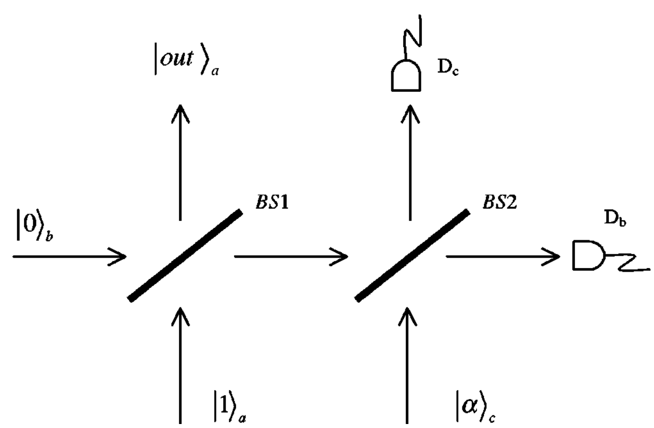

FIG. 1. Sketch of the experimental setup for generation of a traveling wave in our extended superposition of number states. The single-photon fields $|1\rangle_a$ and $|\alpha\rangle_c$ are incident on beam splitters 1 and 2, respectively. When a single photon is detected in D_b and no counts are registered in D_c, then the output state in mode a, $|\text{out}\rangle_a$, is prepared in the desired superposition $C_0|0\rangle + C_1|1\rangle$.

metric beam splitter BS2 (Fig. 1). When a single photon is detected in D_b and no counts are detected in D_c, the field in the output mode a will be projected onto the state $\alpha_0|0\rangle_a + \alpha_1|1\rangle_a$. The projection synthesis of the field mode a will be successfully accomplished if we choose the input state $|\alpha\rangle_c$ such that its vacuum and one-photon amplitudes are proportional to those for the required state, that is, $\alpha_0/C_0 = \alpha_1/C_1$. In this case the state of mode b, before passing though BS2, is given by (up to normalization)

$$ {}_c\langle\alpha|\hat{R}_{bc}^{\dagger}|1\rangle_b|0\rangle_c = \alpha_1^*|0\rangle_b + i\alpha_0^*|1\rangle_b, \tag{3}$$

where

$$\hat{R}_{bc} = \exp[i\theta_2(\hat{b}\hat{c}^{\dagger} + \hat{b}^{\dagger}\hat{c})] \tag{4}$$

is the unitary operator describing the action of BS2. For the case of a 50-50 symmetric BS2 in [8], it follows that $\theta_2 = \pi/4$ and $t_2 = r_2 = 1/\sqrt{2}$.

Here, to extend the previous result for the general case $|\phi\rangle_a = C_0|0\rangle_a + C_1|1\rangle_a + C_2|2\rangle_a + \cdots + C_N|N\rangle_a$, we will assume the scheme sketched in Fig. 1, with the input state in BS1 given by $|\Psi_{in}\rangle_{ab} = |1\rangle_a|N-1\rangle_b$ and the same coherent state incident on BS2 in the mode c, $|\alpha\rangle_c = \Sigma_{n=0}^{\infty}\alpha_n|n\rangle$. We choose the output of BS2 to be $|1\rangle_b$ and $|N-1\rangle_c$. In this way, we have the output of BS1,

$$|\Psi_{out}\rangle_{ab} = \hat{R}_{ab}|\Psi_{in}\rangle_{ab}, \tag{5}$$

whereas, for the whole output,

$$|\Psi_{out}\rangle_{abc} = \hat{R}_{bc}(|\Psi_{out}\rangle_{ab}|\alpha\rangle_c). \tag{6}$$

We stress that for the present generalized quantum scissors θ_1 and θ_2 are free parameters to be adjusted for the achievement of the desired state. The choice made for the detection in D_b and D_c allows us to synthesize the projection leading to the desired output mode a:

$$|\phi\rangle_a = {}_b\langle 1|_c\langle N-1|\Psi_{out}\rangle_{abc}. \tag{7}$$

The foregoing program for calculation of the (desired) state $|\phi\rangle_a$, when applied step by step for $N=1,2,3,\ldots$, allows one to construct, after lengthy calculations, the following recurrence formula (ξ stands for normalization):

$$\begin{aligned}|\phi\rangle_a = \xi\sum_{k=0}^{N}\frac{1}{k![(N-k)!]^{3/2}}\left(-\frac{1}{\eta_1\eta_2}\right)^k \\ \times[(N-k)(1-\delta_{k,N}) - k\eta_2^2(1-\delta_{k,0})] \\ \times[(N-k)(1-\delta_{k,N}) - k\eta_1^2(1-\delta_{k,0})]\alpha^{N-k}|N-k\rangle,\end{aligned} \tag{8}$$

where $\eta_i = r_i/t_i = \sin\theta_i/\cos\theta_i$, $i=1,2$ and, as mentioned before, $r_i(t_i)$ stands for the reflectance (transmittance) of the ith BS. This formula would attain the requirement on the experimental parameters involved to generate the desired superposition state $|\phi\rangle_a = C_0|0\rangle_a + C_1|1\rangle_a + C_2|2\rangle_a + \cdots + C_N|N\rangle_a$. Here we will be concerned with the particular case $C_n = 1$, for all $n=0,1,2,\ldots,N$ (N arbitrary).

Since $k=0,1,2,\ldots,N$, having specified the state to be prepared, Eq. (8) implies a system of N equations. The solubility of such a system can be guaranteed if the number of equations is not greater than the number of free variables. For $N \leqslant 3$ the system is compatible and extends the results found in [8] to $N=2,3$. For $N>3$ the nonlinear system resulting from Eq. (8) will exhibit the same number of free variables (η_1, η_2, α). It will be soluble if $N-3$ equations of this system are redundant, whose occurrence (or not) is not easy to show in view of the nonlinear character of Eq. (8) and approximations coming from the specific numerical methods. However, a by-pass procedure circumventing this difficulty can easily be implemented as follows. When $N>3$ we introduce $N-3$ new parameters, substituting for the field $|\alpha\rangle_c$ entering the BS2 with a (convenient) discrete superposition of coherent states $|\Psi\rangle_c$ [16], all lying over a circle of radius $|\alpha|$ in the complex plane, having different phases, and symmetrically located with respect to the real axis. In this way, e.g., for $N=4$ we employ the auxiliary state

$$\begin{aligned}|\Psi\rangle_c &= \mathcal{N}(|\alpha\rangle_c + |\alpha^*\rangle_c) \\ &= 2\mathcal{N}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_n \frac{|\alpha|^n}{\sqrt{n!}}\cos(n\phi)|n\rangle_c,\end{aligned} \tag{9}$$

where we set $\alpha = |\alpha|\exp(i\phi)$ and $\mathcal{N}$ stands for normalization. The parameter ϕ is the additional parameter to be determined for generation of the truncated phase state $\Sigma_{n=0}^4|n\rangle_c$. When we substitute, in Eq. (6) for the input coherent state $|\alpha\rangle_c$ with the superposition $|\Psi\rangle_c$ given in Eq. (9), the recurrence formula (8) remains valid, changing $|\alpha|^{4-k}$ to $|\alpha|^{4-k}\cos[(4-k)\phi]$. This modification in the auxiliary state in the input mode c leads to the system of equations (for this particular case, $N=4$)

$$(1-3\eta_1^2)(1-3\eta_2^2)|\alpha|\cos\phi = -4\eta_1\eta_2, \tag{10a}$$

$$(1-\eta_1^2)(1-\eta_2^2)|\alpha|^2\cos(2\phi) = 2\sqrt{2}/3, \tag{10b}$$

$$\eta_1\eta_2(3-\eta_1^2)(3-\eta_2^2)|\alpha|^3\cos(3\phi) = -4\sqrt{6}, \tag{10c}$$

$$\eta_1^2\eta_2^2|\alpha|^4\cos(4\phi) = 2\sqrt{6}, \tag{10d}$$

with solution ($\eta_1, \eta_2, |\alpha|, \phi$) given by (0.906, 1.990, 1.477, 4.400).

Extending this strategy for $N=5,6,7,\ldots$ we find that, for arbitrary $N>3$, we must employ the following auxiliary state in the BS2:

$$\begin{aligned}|\Psi\rangle_c &= \mathcal{N}\sum_{m=4}^{N}(|\alpha_m\rangle_c + |\alpha_m^*\rangle_c) \\ &= 2\mathcal{N}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_n\frac{|\alpha|^n}{\sqrt{n!}}\left(\sum_{m=4}^{N}\cos(n\phi_m)\right)|n\rangle_c.\end{aligned} \tag{11}$$

As mentioned above, all the components $\alpha_m = |\alpha|\exp(i\phi_m)$ lie over a circle of radius $|\alpha|$ in the complex plane, having different phases, and symmetrically located with respect to the real axis. The recurrence relation (8) remains valid on changing $|\alpha|^{N-k}$ to $|\alpha|^{N-k}\Sigma_{m=4}^{N}\cos[(N-k)\phi_m]$.

A little inspection of the experimental scheme (Fig. 1) shows that for our choice of selective detection the entire input in BS1 coincides with the entire output in BS2. So the net result of our experimental arrangement is the transformation of the coherent state $|\alpha\rangle_c$ entering BS2 into the wanted state $|\phi\rangle_a$ emerging in BS1. Now, the calculations show that the ingredient directly responsible for this transformation is the state $|\phi\rangle_b$, the output (input) of BS1 (BS2), which connects the coherent state $|\alpha\rangle_c$ and the desired state $|\phi\rangle_a$ being constructed.

As detailed in [8], the practical realization of the *quantum scissors* for the case $C_0|0\rangle_a+C_1|1\rangle_a$ requires a source of a single photon at the first BS (mode b), made to interfere with the coherent field (mode c). This source is obtained from parametric fluorescence [17]. The present scheme, however, would require a source of $N-1$ photons (for engineering a state with a maximum number of photons equal to N). In this case we mention the quantum Fock filter scheme proposed recently by D'Ariano *et al.* [18]: such a scheme is able to select a specific Fock component and superpositions of a few number states from a generic input state. So the scheme proposed in Ref. [18] plays a crucial role in the present generalized quantum scissors device.

Accounting for some sensitive points in the scheme presented above, we first note that the efficiency for single-photon detectors is about 70%, while the damping constant for BS's is considerably smaller, less than 2% in BK7 crystals. Despite the nonunity quantum efficiency of detectors and the absorptive beam splitters, there are other equally important experimental nonidealities. For example, the detectors are required to discriminate between $0,1,2,3,\ldots$ photon arrivals. Since this is currently difficult, an accurate analysis of the errors introduced in engineering a given state should also account for this deficiency.

For completeness, we will consider a pertinent question emerging in the present consideration: what is the nonclassical depth of our truncated state? According to a criterion introduced by Lee [19], the R function, defined as

$$R(z,\tau)=\frac{1}{\tau}\int\frac{d^2w}{\pi}\exp\left(-\frac{1}{\tau}\left|z-w\right|^2\right)P(w), \qquad (12)$$

measures the nonclassical depth of a field state. $P(w)$ is the Glauber-Sudarshan distribution in the coherent representation, satisfying

$$P(w)=\frac{1}{\tau}\int\frac{d^2u}{\pi}\exp[|w|^2+|u|^2+(wu^*-w^*u)]\langle -u|\hat{\rho}|u\rangle, \qquad (13)$$

where $\hat{\rho}$ is the density operator describing a field state and τ is a continuous parameter. For $\tau=0$, 0.5, and 1 the R function coincides with the Glauber, Wigner, and Husimi distributions (P, W, and Q), respectively. Hence, $R(z,\tau)$ interpolates between P and Q, with τ being the interpolating parameter that is used to define the degree of nonclassicality of field states.

Now, since the characteristic of all nonclassical effects is that the P functions of quantum states are singular and not positive definite, the smoothing effect of the convolution transformation is associated with the parameter τ. In fact, the R function becomes acceptable as a classical distribution function (a positive, definite, and regular function) if one considers large enough τ. In this sense, we say that the smoothing operation is complete. The minimum value τ_m of all the τ that complete this operation was adopted in [19] as the nonclassical depth of the quantum state. According to this definition, the unit is an upper bound for τ_m since $R(z;1)=Q(z)=\langle z|\rho|z\rangle$ is always acceptable as a classical distribution function for any quantum state. On the other hand, we have $\tau_m=0$ for an arbitrary coherent state $|\alpha\rangle$, since its P function is given by $\pi\delta^{(2)}(z-\alpha)\geqslant 0$. Hence, the range of τ_m is specified by $\tau_m\in[0,1]$. According to [19], the interval $\tau_m\in(0,1)$ defines a quantum state, whereas $\tau_m=0$ defines a classical state. So the maximum quantum depth is obtained for $\tau_m=1$.

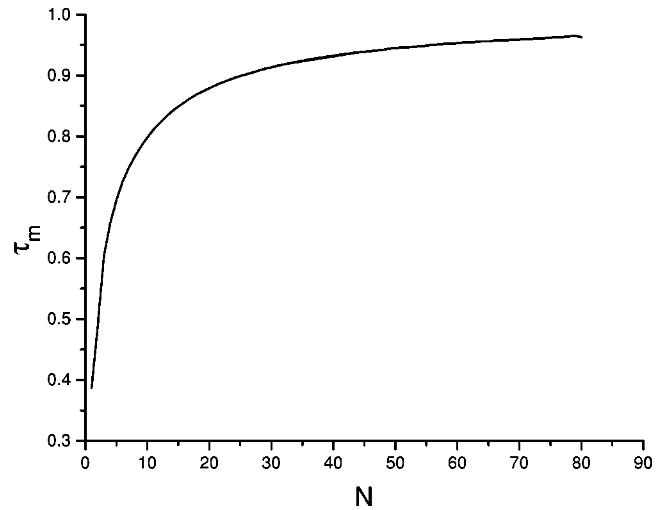

FIG. 2. Plot of the nonclassical depth τ_m as a function of the parameter N, for our truncated state in $(N+1)$-dimensional Hilbert space.

The application of this criterion to our state ($C_n=1,n=0,1,2,\ldots,N$) gives, after lengthy algebra [$\mathcal{L}_n(x)$ stands for Laguerre polynomial],

$$R(z,\tau)=\frac{1}{\tau(N+1)}e^{-|z|^2/\tau}\Bigg\{\sum_{n=0}^{N}\left(-\frac{1-\tau}{\tau}\right)^n\mathcal{L}_n\left(\frac{|z|^2}{\tau(1-\tau)}\right) +2\sum_{n=0}^{N-1}\sum_{l=n+1}^{N}\sqrt{\frac{n!}{l!}}\left(-\frac{1-\tau}{\tau}\right)^n\left(\frac{|z|}{\tau}\right)^{l-n} \times\mathcal{L}_n^{(l-n)}\left(\frac{|z|^2}{\tau(1-\tau)}\right)\Bigg\}. \qquad (14)$$

Figure 2 presents a plot of the nonclassical depth as a function of N, showing that the nonclassicality of this truncated state increases with the dimension of the Hilbert space. A numerical inspection shows that the nonclassical depth tends to unity when $N\to\infty$ (not shown in the figure).

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