Dynamical invariants and nonadiabatic geometric phases in open quantum systems

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We introduce an operational framework to analyze nonadiabatic Abelian and non-Abelian, cyclic and noncyclic, geometric phases in open quantum systems. In order to remove the adiabaticity condition, we generalize the theory of dynamical invariants to the context of open systems evolving under arbitrary convolutionless master equations. Geometric phases are then defined through the Jordan canonical form of the dynamical invariant associated with the superoperator that governs the master equation. As a by-product, we provide a sufficient condition for the robustness of the phase against a given decohering process. We illustrate our results by considering a two-level system in a Markovian interaction with the environment, where we show that the nonadiabatic geometric phase acquired by the system can be constructed in such a way that it is robust against both dephasing and spontaneous emission.

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I. INTRODUCTION

Geometric phases (GPs) provide a remarkable mechanism for a quantum system to keep the memory of its motion as it evolves in Hilbert(-Schmidt) space. These phase factors depend only on the geometry of the path traversed by the system during its evolution. In the context of quantum mechanics, GPs were first obtained by Berry [1], who considered the adiabatic cyclic evolution of a nondegenerate quantum system isolated from the contact with a quantum environment. After the seminal work by Berry, the concept of GPs has been generalized in a number of distinct directions—e.g., degenerate systems [2], nonadiabatic [3] and noncyclic evolutions [4], etc. Besides their conceptual importance in quantum mechanics, GPs have also attracted increasing attention since they were proposed as a tool to achieve fault tolerance in quantum information processing [5,6].

Motivated by the applications in quantum information, a great effort has been devoted to analyzing GPs in *open quantum systems*—i.e., quantum systems subjected to decoherence due to their interaction with a quantum environment [7]. The assumption that a quantum system is closed is always an idealization and, therefore, in order to implement realistic applications in quantum mechanics, we should be able to estimate the effects of the surrounding environment on the dynamics of the system. For a number of physical phenomena, the open system can be conveniently described by a convolutionless (local) master equation after the degrees of freedom of the environment are traced out [7,8]. In this context, several treatments for GPs acquired by the density operator have been proposed (see, e.g., Refs. [9–13]). Moreover, in the particular case of Markovian interaction with the environment, where the system is described by a master equation in the Lindblad form [14], GPs have also been analyzed through quantum trajectories [15–17] (see also Ref. [18] for a further discussion of GPs via stochastic unravelings).

More recently, in the case of adiabatic evolution, Abelian and non-Abelian GPs in open systems have been generally defined in Ref. [19]. This approach was based on an adiabatic approximation previously established for convolutionless master equations [20] (see also Ref. [21] for an application of this adiabatic approximation in adiabatic quantum computation under decoherence and Ref. [22] for an alternative adiabatic approach in weakly coupled open systems). However, although the adiabatic behavior is usually a very welcome feature in theoretical models, it can be unsuitable if decoherence times are small. Therefore, it would be rather desirable to have a general formalism to deal with nonadiabatic GPs for systems under decoherence. In closed systems, a useful tool to remove the adiabaticity constraint of geometric phases [23–26] is the theory of dynamical invariants [27] to treat time-dependent Hamiltonians. Indeed, dynamical invariants were recently used in a proposal of an interferometric experiment to measure nonadiabatic GPs in cavity quantum electrodynamics [28].

The aim of this work is to generalize the theory of dynamical invariants to the context of open quantum systems and to show how this generalization can be used to establish a general approach for nonadiabatic, Abelian and non-Abelian, cyclic and noncyclic, GPs acquired by the components of the density operator of a system evolving under a convolutionless master equation (see also a related work in Ref. [29], which introduced a relationship between GPs and dynamical invariants for a master equation in the Lindblad form). Within our formalism, we will be able to provide a sufficient condition to ensure the robustness of the phase against a given decohering process. As an illustration of our result, we will consider a two-level quantum system (a qubit) interacting with an environment through a Lindblad equa-

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tion. Then, we will find that this system is robust against both dephasing and spontaneous emission. This generalizes the results for the robustness against these decohering processes found in the adiabatic case in Ref. [20]. Furthermore, in the case of spontaneous emission, robustness of the nonadiabatic GP is a feature of our approach, which should positively impact geometric QC (see, e.g., Ref. [17] for difficulties in the correction of spontaneous emission).

II. DYNAMICAL INVARIANTS IN OPEN SYSTEMS

For a closed quantum system, a dynamical invariant $I(t)$ is a Hermitian operator which satisfies [27]

$$\frac{\partial I}{\partial t} - \frac{1}{i}[H,I] = 0, \tag{1}$$

where H is the Hamiltonian of the system. Dynamical invariants have time-independent eigenvalues, implying therefore that their expectation value is constant—i.e., $d\langle I(t)\rangle / dt = 0$.

In order to generalize the concept of a dynamical invariant to the context of open systems, we consider a general open system described by a convolutionless master equation

$$\mathcal{L}\rho = \frac{\partial \rho}{\partial t}, \tag{2}$$

where $\rho(t)$ is the density operator, which can be taken as a vector in Hilbert-Schmidt space, and $\mathcal{L}$ is the (usually non-Hermitian) superoperator which dictates the dynamics of the system. Given an open system governed by $\mathcal{L}(t)$, we define a dynamical invariant $\mathcal{I}(t)$ as a superoperator which satisfies the equation

$$\frac{\partial \mathcal{I}}{\partial t} - [\mathcal{L},\mathcal{I}] = 0. \tag{3}$$

Similarly to the case of closed systems, the eigenvalues of the superoperator $\mathcal{I}(t)$ will be shown to be time independent, as expected for a dynamical invariant. However, note that Eq. (3) does not uniquely determine $\mathcal{I}(t)$ or ensure that such a superoperator exists. The success of our approach will rely therefore on the possibility of constructing nontrivial (time-dependent) dynamical invariants, which can fortunately be found in a number of interesting examples.

The superoperator $\mathcal{I}(t)$ is in general non-Hermitian, which means that it will not always exhibit a basis of eigenstates. However, we can construct a right basis $\{|\mathcal{D}_\alpha^{(i)}\rangle\rangle\}$ and a left basis $\{\langle\langle\mathcal{E}_\alpha^{(i)}|\}$ in Hilbert-Schmidt space based on the Jordan canonical form of $\mathcal{I}(t)$ [30]. Here, the double-ket notation is used to emphasize that these vectors are defined in the space state of linear operators instead of ordinary Hilbert space. This construction is analogous to the procedure developed in Ref. [20], but using now the Jordan decomposition of $\mathcal{I}(t)$ rather than $\mathcal{L}(t)$. It can be shown (see Ref. [20] or Appendix A of Ref. [19]) that left and right basis vectors can always be chosen such that they have the properties

$$\mathcal{I}|\mathcal{D}_\alpha^{(i)}\rangle\rangle = \lambda_\alpha|\mathcal{D}_\alpha^{(i)}\rangle\rangle + |\mathcal{D}_\alpha^{(i-1)}\rangle\rangle, \tag{4}$$

$$\langle\langle\mathcal{E}_\alpha^{(i)}|\mathcal{I} = \langle\langle\mathcal{E}_\alpha^{(i)}|\lambda_\alpha + \langle\langle\mathcal{E}_\alpha^{(i+1)}|, \tag{5}$$

where $|\mathcal{D}_\alpha^{(-1)}\rangle\rangle \equiv 0$ and $\langle\langle\mathcal{E}_\alpha^{(n_\alpha)}| \equiv 0$, with the index α enumerating each Jordan block and the index i enumerating the basis vectors inside each Jordan block, with $i=0,\ldots,n_\alpha-1$ (n_α is the dimension of the block α). Moreover, left and right vectors satisfy the orthonormality condition

$$\langle\langle\mathcal{E}_\alpha^{(i)}|\mathcal{D}_\beta^{(j)}\rangle\rangle = \delta_{\alpha\beta}\delta^{ij}. \tag{6}$$

The eigenvalues of $\mathcal{I}(t)$ are denoted by λ_α, and the left and right eigenvectors of $\mathcal{I}(t)$ are denoted by $|\mathcal{D}_\alpha^{(0)}\rangle\rangle$ and $\langle\langle\mathcal{E}_\alpha^{(n_\alpha-1)}|$, respectively. Taking the derivative of Eq. (4) with respect to time (denoted by the overdot symbol below), we obtain

$$\dot{\mathcal{I}}|\mathcal{D}_\alpha^{(i)}\rangle\rangle + \mathcal{I}|\dot{\mathcal{D}}_\alpha^{(i)}\rangle\rangle = \dot{\lambda}_\alpha|\mathcal{D}_\alpha^{(i)}\rangle\rangle + \lambda_\alpha|\dot{\mathcal{D}}_\alpha^{(i)}\rangle\rangle + |\dot{\mathcal{D}}_\alpha^{(i-1)}\rangle\rangle. \tag{7}$$

Projection of Eq. (7) in $\langle\langle\mathcal{E}_\beta^{(j)}|$ yields

$$\langle\langle\mathcal{E}_\beta^{(j)}|\dot{\mathcal{I}}|\mathcal{D}_\alpha^{(i)}\rangle\rangle = \dot{\lambda}_\alpha\delta_{\alpha\beta}\delta^{ij} + (\lambda_\alpha - \lambda_\beta)\langle\langle\mathcal{E}_\beta^{(j)}|\dot{\mathcal{D}}_\alpha^{(i)}\rangle\rangle + \langle\langle\mathcal{E}_\beta^{(j)}|\dot{\mathcal{D}}_\alpha^{(i-1)}\rangle\rangle - \langle\langle\mathcal{E}_\beta^{(j+1)}|\dot{\mathcal{D}}_\alpha^{(i)}\rangle\rangle. \tag{8}$$

On the other hand, from the definition of a dynamical invariant, given by Eq. (3), we get

$$\langle\langle\mathcal{E}_\beta^{(j)}|\dot{\mathcal{I}}|\mathcal{D}_\alpha^{(i)}\rangle\rangle = (\lambda_\alpha - \lambda_\beta)\langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{L}|\mathcal{D}_\alpha^{(i)}\rangle\rangle + \langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{L}|\mathcal{D}_\alpha^{(i-1)}\rangle\rangle - \langle\langle\mathcal{E}_\beta^{(j+1)}|\mathcal{L}|\mathcal{D}_\alpha^{(i)}\rangle\rangle. \tag{9}$$

By inserting Eq. (9) into Eq. (8), we obtain

$$\dot{\lambda}_\alpha\delta_{\alpha\beta}\delta^{ij} = (\lambda_\alpha - \lambda_\beta)\langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{O}|\mathcal{D}_\alpha^{(i)}\rangle\rangle + \langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{O}|\mathcal{D}_\alpha^{(i-1)}\rangle\rangle - \langle\langle\mathcal{E}_\beta^{(j+1)}|\mathcal{O}|\mathcal{D}_\alpha^{(i)}\rangle\rangle \tag{10}$$

where

$$\mathcal{O} \equiv \mathcal{L} - \frac{\partial}{\partial t}. \tag{11}$$

Let us assume, from now on, that $n_\alpha=1$, i.e., the Jordan blocks are one dimensional (1D). This means that we are assuming that we were able to find a diagonalizable $\mathcal{I}(t)$ (even though it can be non-Hermitian). As we will show below, Abelian GPs will be associated with the situation where $\mathcal{I}(t)$ has nondegenerate 1D Jordan blocks while non-Abelian phases will be associated with the situation where $\mathcal{I}(t)$ displays degenerate 1D Jordan blocks. For multidimensional Jordan blocks, we should proceed with a case-by-case analysis, with no general treatment available.

Therefore, assuming 1D Jordan blocks, we have

$$\dot{\lambda}_\alpha\delta_{\alpha\beta}\delta^{ij} = (\lambda_\alpha - \lambda_\beta)\langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{O}|\mathcal{D}_\alpha^{(i)}\rangle\rangle, \tag{12}$$

where, now, the indices i and j appearing in both $\{|\mathcal{D}_\alpha^{(i)}\rangle\rangle\}$ and $\{\langle\langle\mathcal{E}_\alpha^{(j)}|\}$ account for degenerate states—namely, states such that $\lambda_\alpha=\lambda_\beta$, whichever α and β. Observe that for $\alpha=\beta$ and $i=j$, we obtain $\dot{\lambda}_\alpha=0$, which implies that the dynamical invariant has indeed time-independent eigenvalues. Moreover, taking indices α and β such that $\lambda_\alpha \neq \lambda_\beta$, we obtain

$$\langle\langle\mathcal{E}_\beta^{(j)}|\mathcal{O}|\mathcal{D}_\alpha^{(i)}\rangle\rangle=0 \quad (\lambda_\alpha\neq\lambda_\beta). \tag{13}$$

Equation (13) provides the fundamental condition that will allow for the definition of nonadiabatic GPs.

III. NONADIABATIC GPS VIA DYNAMICAL INVARIANTS

A. Abelian case

Let us assume that the eigenvalues of $\mathcal{I}(t)$ are nondegenerate—i.e., $\lambda_\alpha=\lambda_\beta\Rightarrow\alpha=\beta$. In order to simplify the notation, we will omit the upper index i of the right and left vectors in the Abelian case. Let us take the density operator ρ as a vector in Hilbert-Schmidt space and expand it in the right basis $\{|\mathcal{D}_\alpha\rangle\rangle\}$:

$$|\rho\rangle\rangle=\sum_\alpha c_\alpha|\mathcal{D}_\alpha\rangle\rangle. \tag{14}$$

By inserting Eq. (14) into the master equation (2) and projecting it in $\langle\langle\mathcal{E}_\beta|$, we obtain

$$\dot{c}_\beta=\sum_\alpha c_\alpha\langle\langle\mathcal{E}_\beta|\mathcal{O}|\mathcal{D}_\alpha\rangle\rangle. \tag{15}$$

By using Eq. (13), we can get rid of the sum in Eq. (15), which implies

$$\dot{c}_\beta=c_\beta\langle\langle\mathcal{E}_\beta|\mathcal{O}|\mathcal{D}_\beta\rangle\rangle. \tag{16}$$

Solving Eq. (16), we obtain

$$c_\beta(t)=c_\beta(0)\exp\left(-\int_0^t\left\langle\left\langle\mathcal{E}_\beta\left|\frac{\partial}{\partial t'}\right|\mathcal{D}_\beta\right\rangle\right\rangle dt'\right)\times\exp\left(\int_0^t\langle\langle\mathcal{E}_\beta|\mathcal{L}|\mathcal{D}_\beta\rangle\rangle dt'\right). \tag{17}$$

Therefore, each right eigenvector $|\mathcal{D}_\beta\rangle\rangle$ in the expansion of ρ gets multiplied by a phase. The first exponential in Eq. (17) gives an origin to the geometric contribution of the phase whereas the second exponential generates the dynamical sector. The geometric phase must be gauge invariant, i.e., it cannot be modified (or eliminated) by a multiplication of the basis vectors $\{|\mathcal{D}_\alpha\rangle\rangle\}$ or $\{\langle\langle\mathcal{E}_\alpha|\}$ by a local (time-dependent) complex factor. Indeed, let us consider the redefinition $|\mathcal{D}'_\alpha\rangle\rangle=\chi(t)e^{i\nu(t)}|\mathcal{D}_\alpha\rangle\rangle$ $[\chi(t)\neq 0,\ \forall\ t]$. For the left vectors, the orthonormality condition, given by Eq. (6), imposes that $\langle\langle\mathcal{E}'_\alpha|=\langle\langle\mathcal{E}_\alpha|\chi^{-1}e^{-i\nu(t)}$. Gauge invariance under these transformations for an arbitrary (cyclic or noncyclic) path in Hilbert-Schmidt space is achieved by adding a new term in the expression of the GP in Eq. (17), resulting in

$$\varphi_\beta=\ln[\langle\langle\mathcal{E}_\beta(0)|\mathcal{D}_\beta(t)\rangle\rangle]-\int_0^t\left\langle\left\langle\mathcal{E}_\beta(t')\left|\frac{\partial}{\partial t'}\right|\mathcal{D}_\beta(t')\right\rangle\right\rangle dt'. \tag{18}$$

By a direct inspection, it can be shown that φ_β is gauge invariant. This is analogous to the procedure used in Ref. [4] to extend Berry phases for noncyclic paths in closed systems. The contribution coming from the term $\ln[\langle\langle\mathcal{E}_\beta(0)|\mathcal{D}_\beta(t)\rangle\rangle]$ may affect the visibility of the phase, since $\langle\langle\mathcal{E}_\beta(0)|\mathcal{D}_\beta(t)\rangle\rangle$ is not necessarily a complex number with modulus 1. Moreover, note that for a cyclic path of the basis vectors—i.e., $|\mathcal{D}_\alpha(t)\rangle\rangle=|\mathcal{D}_\alpha(0)\rangle\rangle$—we have $\ln[\langle\langle\mathcal{E}_\beta(0)|\mathcal{D}_\beta(0)\rangle\rangle]=\ln 1=0$. Therefore, for the cyclic GP, no extra term should be added, with φ_β simplifying to

$$\varphi_\beta^{\text{cyclic}}=-\int_0^t\left\langle\left\langle\mathcal{E}_\beta(t')\left|\frac{\partial}{\partial t'}\right|\mathcal{D}_\beta(t')\right\rangle\right\rangle dt' \quad \text{(cyclic path)}. \tag{19}$$

Observe also that the phases defined above are nonadiabatic, since no adiabaticity requirement has been imposed in any step of our derivation.

B. Non-Abelian case

Let us consider now the case of 1D degenerate Jordan blocks and expand the density operator as

$$|\rho\rangle\rangle=\sum_{\alpha=1}^m\sum_j c_\alpha^{(j)}|\mathcal{D}_\alpha^{(j)}\rangle\rangle, \tag{20}$$

where m is the number of Jordan blocks and j identifies all the right eigenvectors $|\mathcal{D}_\alpha^{(j)}\rangle\rangle$ of $\mathcal{I}(t)$ associated with the eigenvalue λ_α. Similarly as in the nondegenerate case, we insert Eq. (20) into Eq. (2) and project the result in $\langle\langle\mathcal{E}_\beta^{(i)}|$, yielding

$$\dot{c}_\beta^{(i)}=\sum_{\alpha=1}^m\sum_j c_\alpha^{(j)}\langle\langle\mathcal{E}_\beta^{(i)}|\mathcal{O}|\mathcal{D}_\alpha^{(j)}\rangle\rangle. \tag{21}$$

By making use of Eq. (13), we obtain

$$\dot{c}_\beta^{(i)}=\sum_j c_\beta^{(j)}\langle\langle\mathcal{E}_\beta^{(i)}|\mathcal{O}|\mathcal{D}_\beta^{(j)}\rangle\rangle. \tag{22}$$

Now let us define the matrix M_β, whose elements are given by

$$M_\beta^{(ij)}=\langle\langle\mathcal{E}_\beta^{(i)}|\mathcal{O}|\mathcal{D}_\beta^{(j)}\rangle\rangle=H_\beta^{(ij)}+A_\beta^{(ij)}, \tag{23}$$

with

$$H_\beta^{(ij)}=\langle\langle\mathcal{E}_\beta^{(i)}|\mathcal{L}|\mathcal{D}_\beta^{(j)}\rangle\rangle,$$
$$A_\beta^{(ij)}=-\left\langle\left\langle\mathcal{E}_\beta^{(i)}\left|\frac{\partial}{\partial t}\right|\mathcal{D}_\beta^{(j)}\right\rangle\right\rangle. \tag{24}$$

Note that H_β plays the role of a non-Abelian dynamical phase while A_β will correspond to a geometrical contribution to the total phase. Moreover, by defining the vector $\vec{\mathbf{c}}_\beta=(c_\beta^1,\ldots,c_\beta^N)^t$ in Hilbert-Schmidt space, with N denoting the degree of degeneracy, we get

$$\dot{\vec{\mathbf{c}}}_\beta=M_\beta\vec{\mathbf{c}}_\beta, \tag{25}$$

whose formal solution is

$$\vec{\mathbf{c}}_\beta(t)=\mathcal{U}_\beta\vec{\mathbf{c}}_\beta(0), \tag{26}$$

with

$$\mathcal{U}_\beta = \mathcal{T} \exp\left[\int_0^t [H_\beta(t') + A_\beta(t')]dt'\right], \quad (27)$$

where $\mathcal{T}$ is the time-ordering operator. It is important to note that the matrices H_β and A_β do not commute in general. This means that, in the non-Abelian case, the dynamical and GPs may not be easily split up. This is indeed a feature which also appears in closed systems for nonadiabatic non-Abelian phases [25,26,31]. By assuming that $[\int A_\beta(t)dt, \int H_\beta(t)dt]=0$, we can extend the reasoning in Ref. [29] for Hilbert-Schmidt space, with the noncyclic non-Abelian GP getting the form

$$\exp \Phi_\beta = \mathcal{W}_\beta(t) \mathcal{T} \exp\left[\int_0^t A_\beta(t')dt'\right],$$

where $\mathcal{W}_\beta$ is the overlap matrix, whose elements are given by $\mathcal{W}_\beta^{(ij)}(t) = \langle\langle \mathcal{E}_\beta^{(i)}(0) | \mathcal{D}_\beta^{(j)}(t)\rangle\rangle$. The presence of the overlap matrix ensures gauge invariance of the non-Abelian GP, which can be verified by a similar inspection as that discussed in Sec. III A. Moreover, note that $\mathcal{W}_\beta$ reduces to the identity for cyclic evolutions.

C. Adiabatic limit

Let us turn now to an observation about the adiabatic regime. The GPs defined in the previous sections will get reduced to the adiabatic case introduced in Ref. [19] for the choice of a slowly varying dynamical invariant. Indeed, let us suppose that

$$\frac{\partial \mathcal{I}}{\partial t} \approx 0. \quad (28)$$

In this case, by taking into account Eq. (3), we will obtain $[\mathcal{L}, \mathcal{I}] \approx 0$. Then, by assuming that both $\mathcal{L}$ and $\mathcal{I}$ are diagonalizable, it follows that they will have a common basis of eigenstates. Therefore, under the condition (28), the nonadiabatic basis, given by eigenstates of $\mathcal{I}$, will exactly be the same as the adiabatic basis, given by the eigenstates of $\mathcal{L}$.

IV. NONADIABATIC GP FOR A TWO-LEVEL SYSTEM UNDER DECOHERENCE

Let us examine the GP acquired by a two-level system described by the free Hamiltonian

$$H = \frac{\omega}{2}\sigma_z. \quad (29)$$

Under decoherence in a Markovian environment, the dynamics of the system will be governed by the Lindblad equation [14]

$$\frac{\partial \rho}{\partial t} = -i[H,\rho] - \frac{1}{2}\sum_i (\Gamma_i^\dagger \Gamma_i \rho + \rho \Gamma_i^\dagger \Gamma_i - 2\Gamma_i \rho \Gamma_i^\dagger). \quad (30)$$

A. Robustness under dephasing

Let us start by taking the case of dephasing, where $\Gamma(t) = \gamma_d \sigma_z$. In this case, the superoperator $\mathcal{L}$ can be written as (see the Appendix)

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma_d^2 & -\omega & 0 \\ 0 & \omega & -2\gamma_d^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (31)$$

Therefore, $\mathcal{L}$ has a 2×2 matrix representation given by

$$\mathcal{L} = \begin{pmatrix} -2\gamma_d^2 & -\omega \\ \omega & -2\gamma_d^2 \end{pmatrix}. \quad (32)$$

Let us look for a simple nontrivial superoperator $\mathcal{I}(t)$, which we propose to take the form

$$\mathcal{I} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}, \quad (33)$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are time-dependent well-behaved functions. Now, it follows an important fact about the robustness of the nonadiabatic GP. For arbitrary time-dependent functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$, we have that the commutator $[\mathcal{L}, \mathcal{I}]$ is independent of the dephasing parameter γ_d. Indeed,

$$[\mathcal{L}, \mathcal{I}] = \omega \begin{pmatrix} -\beta-\gamma & \alpha-\delta \\ \alpha-\delta & \beta+\gamma \end{pmatrix}. \quad (34)$$

Due to this property, we can construct a nontrivial (time-dependent) superoperator $\mathcal{I}(t)$ that is independent of γ_d. This operator will generate right and left bases which are also independent of γ_d. Hence the GP acquired by the density operator ρ will keep the independence of γ_d, exhibiting therefore robustness against dephasing.

Let us analyze in detail the GP acquired during a cyclic path of the left and right vectors. By imposing Eq. (3), we will get a set of coupled differential equations

$$\dot{\alpha} = -\omega(\beta+\gamma),$$

$$\dot{\beta} = \omega(\alpha-\delta),$$

$$\dot{\gamma} = \omega(\alpha-\delta),$$

$$\dot{\delta} = \omega(\beta+\gamma). \quad (35)$$

The solution of this set of equations yields

$$\mathcal{I} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t)+c_2 & -\alpha(t)+c_1 \end{pmatrix}, \quad (36)$$

where

$$\alpha(t) = \alpha_1 \cos 2\omega t + \alpha_2 \sin 2\omega t + \frac{c_1}{2},$$

$$\beta(t) = \alpha_1 \sin 2\omega t - \alpha_2 \cos 2\omega t - \frac{c_2}{2}, \quad (37)$$

with α_1, α_2, c_1, and c_2 denoting arbitrary constants. Therefore, as mentioned before, we can construct the dynamical invariant such that it is independent of γ_d. The superoperator

$\mathcal{I}(t)$ given in Eq. (36) has a basis of eigenvectors as long as $4(\alpha_1^2+\alpha_2^2)\neq c_2^2$. This can be adjusted with no problem since we are free to set the constants. The operator $\mathcal{I}(t)$ is in general non-Hermitian, which means that the left and right bases will not be related by a transpose conjugation operation. The cyclic GPs φ_1 and φ_2 associated with the right vectors $|\mathcal{D}_1\rangle\rangle$ and $|\mathcal{D}_2\rangle\rangle$, respectively, can be computed as given by Eq. (19), yielding

$$\varphi_1 = -\int_0^t \left\langle\left\langle \mathcal{E}_1 \left| \frac{\partial}{\partial t'} \right| \mathcal{D}_1 \right\rangle\right\rangle dt', \tag{38}$$

$$\varphi_2 = -\int_0^t \left\langle\left\langle \mathcal{E}_2 \left| \frac{\partial}{\partial t'} \right| \mathcal{D}_2 \right\rangle\right\rangle dt'. \tag{39}$$

Indeed, by choosing a cyclic path for the basis vectors, we set $t=2\pi/\omega$. Therefore, we obtain

$$\varphi_1 = -2\pi \frac{c_2 v_1 + 2v_2\sqrt{-(v_1/v_3)^2}}{v_1 v_3},$$

$$\varphi_2 = -\varphi_1, \tag{40}$$

where $v_1\equiv 2\alpha_2+c_2$, $v_2\equiv\alpha_1^2+\alpha_2^2$, and $v_3\equiv\sqrt{4v_2-c_2^2}$. Note that the GP depends on the particular choice of the superoperator $\mathcal{I}(t)$, since it depends on the values of α_1, α_2, and c_2. Indeed, different choices of $\mathcal{I}(t)$ will imply in distinct right and left bases. An interesting particular case is the choice $c_2=0$. In this situation, we obtain

$$\varphi_1 = -i\pi\frac{|\alpha_2|}{\alpha_2} = -i\pi\,\mathrm{sgn}(\alpha_2), \tag{41}$$

$$\varphi_2 = +i\pi\frac{|\alpha_2|}{\alpha_2} = +i\pi\,\mathrm{sgn}(\alpha_2). \tag{42}$$

Note that, besides robustness against dephasing, the GPs given by Eqs. (41) and (42) display only an oscillating (imaginary) term. The loss of visibility typical in open systems, which is given by the presence of damping real exponentials, is absent for the GP in the case $c_2=0$. Naturally, a loss of visibility may still come (and indeed it does) from the dynamical phase.

B. Robustness under spontaneous emission

Now let us analyze the robustness of the GP against spontaneous emission, which is modeled by $\Gamma=\gamma_{se}\sigma_-$, with $\sigma_-=\sigma_x-i\sigma_y$. In this case, the Lindblad superoperator is given by (see the Appendix)

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma_{se}^2 & -\omega & 0 \\ 0 & \omega & -2\gamma_{se}^2 & 0 \\ 4\gamma_{se}^2 & 0 & 0 & -4\gamma_{se}^2 \end{pmatrix}. \tag{43}$$

The superoperator $\mathcal{L}$ motivates the proposal of the dynamical invariant

$$\mathcal{I}(t) = \begin{pmatrix} q(t) & 0 & 0 & p(t) \\ 0 & \alpha(t) & \beta(t) & 0 \\ 0 & \gamma(t) & \delta(t) & 0 \\ x(t) & 0 & 0 & y(t) \end{pmatrix}, \tag{44}$$

where the matrix elements are arbitrary time-dependent functions. The commutator $[\mathcal{L},\mathcal{I}]$ is now given by

$$[\mathcal{L},\mathcal{I}] = \begin{pmatrix} 4\gamma_{se}^2 p & 0 & 0 & 4\gamma_{se}^2 p \\ 0 & -\varepsilon\omega & \eta\omega & 0 \\ 0 & \eta\omega & \varepsilon\omega & 0 \\ -4\gamma_{se}^2(q+x-y) & 0 & 0 & -4\gamma_{se}^2 p \end{pmatrix}, \tag{45}$$

where $\varepsilon=\beta+\gamma$ and $\eta=\alpha-\delta$. We observe that the commutator is split out in two submatrices. The internal submatrix is identical to that obtained from dephasing [see Eq. (34)], being independent of the decoherence parameter γ_{se}. In order to ensure robustness for the external submatrix, we must impose $p=0$ [implying from Eq. (3) that both q and y are constants] and $q=y-x$ (implying that x is also a constant). Since, as given by Eq. (44), the internal and external submatrices are decoupled, only the internal submatrix will contribute for the GP (the constant elements of the external submatrix will disappear in the computation of the GP, due to the time derivative). This means that (i) the invariant superoperator $\mathcal{I}(t)$ for spontaneous emission given by Eq. (44) will produce the same GP as that obtained for dephasing; (ii) since $\mathcal{I}(t)$ can be nontrivially defined as independent of γ_{se}, then the nonadiabatic GP acquired by ρ in the basis of $\mathcal{I}(t)$ is robust against spontaneous emission. The robustness of the geometric phase under spontaneous emission appears here as a consequence of the expansion of the density operator ρ in the basis of a suitably chosen invariant superoperator (see, e.g., Ref. [16] for an analysis based on quantum trajectories of a geometric phase which is nonrobust against spontaneous emission).

C. Example of nonrobustness: Bit flip

Robustness will not be present for arbitrary processes. For instance, consider the case of bit flip—i.e., $\Gamma=\gamma_b\sigma_x$. In this case, the Lindblad superoperator reads (see the Appendix)

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & \omega & -2\gamma_b^2 & 0 \\ 0 & 0 & 0 & -2\gamma_b^2 \end{pmatrix}. \tag{46}$$

Consider that we propose the dynamical invariant $\mathcal{I}$ given by Eq. (44). The commutator $[\mathcal{L},\mathcal{I}]$ now yields

$$[\mathcal{L},\mathcal{I}] = \begin{pmatrix} 0 & 0 & 0 & 2\gamma_b^2 p \\ 0 & -\varepsilon\omega & 2\beta\gamma_b^2+\eta\omega & 0 \\ 0 & -2\gamma\gamma_b^2+\eta\omega & \varepsilon\omega & 0 \\ -2\gamma_b^2 x & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

where, as defined for the case of spontaneous emission, $\varepsilon = \beta+\gamma$ and $\eta=\alpha-\delta$. Therefore, the requirement of independence of γ_b yields $x=0$, $p=0$, $\omega(\alpha-\delta)=-2\beta\gamma_b^2$, and $\omega(\alpha-\delta)=-2\gamma\gamma_b^2$. Then, by using Eqs. (37), we obtain $\alpha = c_1/2$ and $\beta=-c_2/4$ which, from Eq. (36), imply that α, β, γ, and δ are constants. Moreover, requiring Eq. (3) for the dynamical invariant, we also find that q and y are constants. Therefore $\mathcal{I}$ as given by Eq. (44) cannot result in nonvanishing GPs which are robust against bit flip, since the robust dynamical invariant obtained is trivially constant. Thus, let us turn to the case of time-dependent $\mathcal{I}(t)$ and explicitly analyze the dependence of the geometric phase on the parameter γ_b. By taking $x=0$ and $p=0$ in Eq. (47), we can choose the dynamical invariant as

$$\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha(t) & \beta(t) & 0 \\ 0 & \gamma(t) & \delta(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{48}$$

where now the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ satisfy the following set of differential equations:

$$\dot{\alpha} = -(\beta+\gamma)\omega, \tag{49}$$

$$\dot{\beta} = 2\beta\gamma_b^2 + (\alpha-\delta)\omega, \tag{50}$$

$$\dot{\gamma} = -2\gamma\gamma_b^2 + (\alpha-\delta)\omega, \tag{51}$$

$$\dot{\delta} = (\beta+\gamma)\omega. \tag{52}$$

The solution of Eqs. (49)–(52) can be written as

$$\alpha(t) = \omega\frac{(-\varepsilon_1 e^{2\xi t}+\varepsilon_2 e^{-2\xi t})}{2\xi} + \alpha_1,$$

$$\beta(t) = \frac{\varepsilon(t)+\sigma(t)}{2},$$

$$\gamma(t) = \frac{\varepsilon(t)-\sigma(t)}{2},$$

$$\delta(t) = -\alpha(t) + c_1, \tag{53}$$

where α_1, ε_1, ε_2, and c_1 are constants, $\xi=(\gamma_b^4-\omega^2)^{1/2}$, and

$$\varepsilon(t) = \varepsilon_1 e^{2\xi t} + \varepsilon_2 e^{-2\xi t},$$

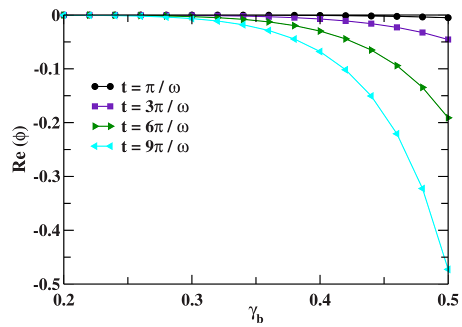

FIG. 1. (Color online) Real part of the geometric phase for a two-level system under bit flip as a function of the decoherence parameter γ_b (in units such that $\omega=1$).

$$\sigma(t) = \gamma_b^2\left(\frac{\varepsilon_1 e^{2\xi t}-\varepsilon_2 e^{-2\xi t}}{\xi}\right) + \sigma_1, \tag{54}$$

with σ_1 satisfying $\gamma_b^2\sigma_1+\omega(2\alpha_1-c_1)=0$. We are free to set the initial conditions which define the dynamical invariant $\mathcal{I}(t)$. Distinct choices of $\mathcal{I}(t)$ will imply different GPs acquired by the basis vectors $|\mathcal{D}_\alpha^{(i)}\rangle\rangle$ that expand the density operator. In order to consider a concrete example, we set $\sigma_1=0$, which implies $c_1=2\alpha_1$. Moreover, we take $\varepsilon_1=-0.5$ and $\varepsilon_2=1$. By adopting these values, we plot in Fig. 1 the real part of the GP ϕ, given by Eq. (18), as a function of the decoherence parameter for several fixed times.

This GP is noncyclic and evaluated for the eigenstate of $\mathcal{I}$ associated with the eigenvalue $\alpha_1-\sqrt{\varepsilon_1\varepsilon_2}$ (the GP is independent of α_1). Note that the visibility of ϕ decreases faster as we increase the evolution time t. Concerning the imaginary part of ϕ, it can be shown that it is independent of γ_b for a given time t.

We can also consider the dependence of the GP as time is varied for a fixed γ_b. This is plotted in Figs. 2 and 3, where we fix $\gamma_b=0.1$ (in units such that $\omega=1$). As we can observe in Fig. 2, the imaginary part of the gauge-invariant GP, which is the sum of ϕ^{cyclic} [see Eq. (19)] and the logarithmic correction, behaves as a step function of time. The origin of this behavior is the ln term in Eq. (18). Moreover, note that the discontinuities in the imaginary part of the GP are asso-

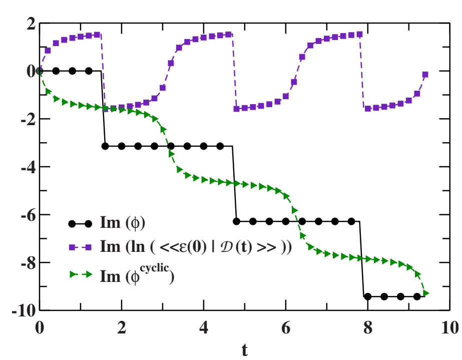

FIG. 2. (Color online) Imaginary part of the GP for a two-level system under bit flip as a function of time. The decoherence parameter γ_b is set to 0.1 (in units such that $\omega=1$).

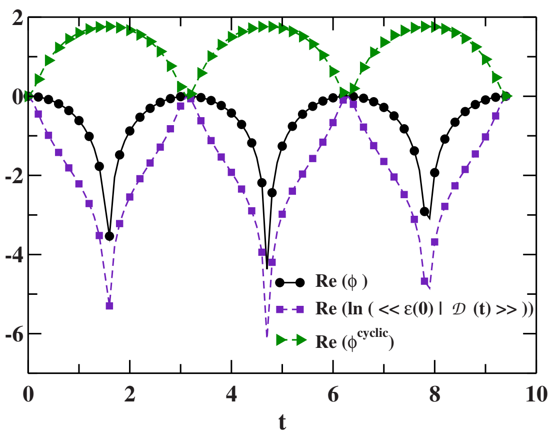

FIG. 3. (Color online) Real part of the GP for a two-level system under bit flip as a function of time. The decoherence parameter γ_b is set to 0.1 (in units such that $\omega=1$).

ciated with a pronounced behavior also in the real part, as exhibited in Fig. 3.

D. Dynamical phase under decoherence

Concerning the behavior of the dynamical phase, it will usually not exhibit robustness against decoherence. This is due to the fact that the superoperator $\mathcal{L}$ depends on the decoherence parameters. This is in contrast with the invariant superoperator $\mathcal{I}$, which can be designed to display robustness if $[\mathcal{L},\mathcal{I}]$ is independent of the decohering processes (as previously shown for dephasing and spontaneous emission). Indeed, robustness of the dynamical phase can only be achieved whether the integral $\int\langle\langle\mathcal{E}_\beta|\mathcal{L}|\mathcal{D}_\beta\rangle\rangle dt'$ can be made independent of decoherence, which turns out to be a nongeneric situation. As a concrete example, let us consider the dynamical phase for dephasing. In this case, robustness is not possible by choosing the invariant operator given in Sec. IV A. In fact, by explicit computation for a cyclic evolution, we obtain

$$\int_0^{2\pi/\omega}\langle\langle\mathcal{E}_1|\mathcal{L}|\mathcal{D}_1\rangle\rangle dt' = -\frac{4\pi}{\omega}\gamma_d^2 + \frac{2c_2\pi}{v_3}, \qquad (55)$$

$$\int_0^{2\pi/\omega}\langle\langle\mathcal{E}_2|\mathcal{L}|\mathcal{D}_2\rangle\rangle dt' = -\frac{4\pi}{\omega}\gamma_d^2 - \frac{2c_2\pi}{v_3}, \qquad (56)$$

with v_3 defined as in Eq. (40). Therefore, notice that no adjust can be done in order to remove the dependence of the dynamical phase for an arbitrary γ_d. As expected, this dependence will induce a damping contribution to the visibility of the total phase.

V. CONCLUSIONS

We have proposed a generalization of the theory of dynamical invariants to the context of open quantum systems. This approach can be seen as an alternative way to solve the master equation, since the construction and diagonalization of a dynamical invariant automatically determines the density operator. By using this generalization, we have defined in general nonadiabatic GPs acquired by the density operator during its evolution in Hilbert-Schmidt space. Moreover, we have delineated a strategy to look for nonadiabatic GPs that are robust against a given decoherence process. Our method consists in looking for dynamical invariants such that $[\mathcal{L},\mathcal{I}]$ is independent of the decohering parameters. As an illustration of our approach, we have analyzed the GP acquired by a qubit evolving under decoherence. The GP in this case was shown to be robust against both dephasing and spontaneous emission. Robustness of the nonadiabatic GP against spontaneous emission is a remarkable feature which may have a positive impact in geometric quantum computation. In this direction, a certainly interesting application of our approach is the analysis of non-Abelian geometric phases in the tripod-linkage system of atomic states [32–35]. We left this topic for further research.

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APPENDIX: LINDBLAD SUPEROPERATOR FOR A TWO-LEVEL SYSTEM UNDER DECOHERENCE

Let us illustrate the construction of the Lindblad superoperator $\mathcal{L}$ by examining a two-level system described by the free Hamiltonian given by Eq. (29). We will consider the following decohering process:

$$\Gamma(t) = \alpha_1(t)\sigma_x + \alpha_2(t)\sigma_y + \alpha_3(t)\sigma_z = \sum_{i=1}^{3}\alpha_i(t)\sigma_i, \qquad (A1)$$

where $\sigma_1\equiv\sigma_x$, $\sigma_2\equiv\sigma_y$, and $\sigma_3\equiv\sigma_z$. Note that $\Gamma(t)$ describes an arbitrary single decoherence process for a two-level system. For instance, for dephasing, we would take $\alpha_1=\alpha_2=0$. For the density operator, we can take the expression

$$\rho(t) = \frac{1}{2}(I+\vec{v}\cdot\vec{\sigma}) = \frac{1}{2}(I+v_1\sigma_x+v_2\sigma_y+v_3\sigma_z), \qquad (A2)$$

where I is the two-dimensional identity operator and $\vec{v}$ is the coherence vector. By inserting Eqs. (29), (A1), and (A2) into the Lindblad equation (30) we obtain

$$\frac{\partial\rho}{\partial t} = \frac{\omega}{2}(v_1\sigma_2 - v_2\sigma_1) + \sum_{i,j}\frac{\alpha_i^\dagger\alpha_j}{2}(v_i\sigma_j + v_j\sigma_i) - \sum_{i,j}|\alpha_i|^2 v_j\sigma_j - \sum_{i,j,k} i\varepsilon_{ijk}\alpha_i^\dagger\alpha_j\sigma_k, \qquad (A3)$$

where we have made use of the auxiliary expressions

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}I, \quad \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \qquad (A4)$$

with the repeated indices k summed over and with ε_{ijk} denoting the Levi-Cività symbol [it is 1 if (i,j,k) is an even permutation of (1,2,3), −1 if it is an odd permutation, and 0 if any index is repeated]. Factoring out the components in each σ_i direction, Eq. (A3) can be rewritten as

$$\frac{\partial\rho}{\partial t} = \left[-\frac{\omega v_2}{2} + (\alpha_1^\dagger\alpha_2 + \alpha_1\alpha_2^\dagger)\frac{v_2}{2} + (\alpha_1^\dagger\alpha_3 + \alpha_1\alpha_3^\dagger)\frac{v_3}{2} - (|\alpha_2|^2 + |\alpha_3|^2)v_1 + i(\alpha_2^\dagger\alpha_3 - \alpha_2\alpha_3^\dagger)\right]\sigma_1 + \left[\frac{\omega v_1}{2} + (\alpha_1^\dagger\alpha_2 + \alpha_1\alpha_2^\dagger)\frac{v_1}{2} + (\alpha_2^\dagger\alpha_3 + \alpha_2\alpha_3^\dagger)\frac{v_3}{2} - (|\alpha_1|^2 + |\alpha_3|^2)v_2 - i(\alpha_1^\dagger\alpha_3 - \alpha_1\alpha_3^\dagger)\right]\sigma_2 + \left[(\alpha_1^\dagger\alpha_3 + \alpha_1\alpha_3^\dagger)\frac{v_1}{2} + (\alpha_2^\dagger\alpha_3 + \alpha_2\alpha_3^\dagger)\frac{v_2}{2} - (|\alpha_1|^2 + |\alpha_2|^2)v_3 + i(\alpha_1^\dagger\alpha_2 - \alpha_1\alpha_2^\dagger)\right]\sigma_3. \tag{A5}$$

Taking $\rho(t)$ as a vector in Hilbert-Schmidt space and using Eq. (A2), we can write

$$|\rho(t)\rangle\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{A6}$$

where $|\rho(t)\rangle\rangle$ is expressed in the basis $\{I,\sigma_1,\sigma_2,\sigma_3\}$. Therefore, by inserting Eqs. (A6) and (A5) (for $\frac{\partial\rho}{\partial t}$) into Eq. (2), we obtain the Lindblad superoperator $\mathcal{L}$:

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2i(\alpha_2^\dagger\alpha_3 - \alpha_2\alpha_3^\dagger) & -2(|\alpha_2|^2 + |\alpha_3|^2) & -\omega + (\alpha_1^\dagger\alpha_2 + \alpha_1\alpha_2^\dagger) & (\alpha_1^\dagger\alpha_3 + \alpha_1\alpha_3^\dagger) \\ -2i(\alpha_1^\dagger\alpha_3 - \alpha_1\alpha_3^\dagger) & \omega + (\alpha_1^\dagger\alpha_2 + \alpha_1\alpha_2^\dagger) & -2(|\alpha_1|^2 + |\alpha_3|^2) & (\alpha_2^\dagger\alpha_3 + \alpha_2\alpha_3^\dagger) \\ 2i(\alpha_1^\dagger\alpha_2 - \alpha_1\alpha_2^\dagger) & (\alpha_1^\dagger\alpha_3 + \alpha_1\alpha_3^\dagger) & (\alpha_2^\dagger\alpha_3 + \alpha_2\alpha_3^\dagger) & -2(|\alpha_1|^2 + |\alpha_2|^2) \end{pmatrix}. \tag{A7}$$

Some interesting particular cases of Eq. (A7) can be obtained. For instance, for dephasing, we have $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \equiv \gamma_d$, resulting in Eq. (31). Note that the first column of $\mathcal{L}$ vanishes for dephasing. In fact, this will be the case whenever the parameters α_i are real. An interesting case of complex α_i is given by spontaneous emission, where $\alpha_1 \equiv \gamma$, $\alpha_2 \equiv -i\gamma$, and $\alpha_3 = 0$. In this case, we obtain the superoperator $\mathcal{L}$ shown in Eq. (43).

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