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Quantum coupled oscillators versus forced oscillator

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Abstract
We compare the time evolutions of two quantum systems: a quantum harmonic oscillator driven by a *classical force* and two coupled quantum harmonic oscillators. We find the conditions under which the first system can be considered as a 'mean-field approximation' to the second one. This happens in the weak coupling limit provided that the 'master' oscillator was initially in a highly excited quantum state with small fluctuations of the canonical variables (such as the coherent state or displaced thermal state), while the 'subordinate' oscillator must not be 'too excited'. The essential difference between the resonance and nonresonance cases is discussed. In particular, the effect of the resonance-quantum-state exchange is demonstrated.

Keywords: Coupled oscillators, quantum forced oscillator, quantum state exchange, coherent states, displaced thermal states

1. Introduction

The quantum forced-oscillator model has been intensively studied over the decades from different viewpoints. Solutions of the Schrödinger equation with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + \frac{m\omega^2}{2}x_1^2 - f(t)x_1 \tag{1}$$

(the so-called quantum forced oscillator) have been known since the 1950s [1–4]. In this case, the dynamics is determined by the 'time-dependent Fourier transform' of the force $f(t)$:

$$\delta(t) = -\frac{\mathrm{i}}{\sqrt{2\hbar m\omega}}\int_0^t \exp(\mathrm{i}\omega\tau)\, f(\tau)\,\mathrm{d}\tau \equiv \frac{\delta_2 + \mathrm{i}\delta_1}{\sqrt{2\hbar m\omega}} \tag{2}$$

or by its (scaled) real and imaginary parts

$$\delta_1(t) = -\int_0^t f(\tau)\cos(\omega\tau)\,\mathrm{d}\tau, \tag{3}$$

$$\delta_2(t) = \int_0^t f(\tau)\sin(\omega\tau)\,\mathrm{d}\tau. \tag{4}$$

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For instance, these functions enter the propagator [5, 6]

$$G_f(x_1, x_1', t) = \left(\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}\right)^{1/2} \times \exp\left\{\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\left[\cos\omega t\left(x_1^2 + x_1'^2\right) - 2x_1x_1'\right] + \frac{\mathrm{i}}{\hbar}\left[x_1 g(t) - x_1'\tilde{g}(t) + \phi(t)\right]\right\} \tag{5}$$

through the combinations

$$g(t) = \frac{\delta_2(t)}{\sin\omega t}, \qquad \tilde{g}(t) = \delta_1(t) + \delta_2(t)\cot\omega t, \tag{6}$$

$$\phi(t) = (2m\omega)^{-1}\left(\delta_2^2(t)\cot\omega t + 2\int_0^t \frac{\mathrm{d}\delta_1}{\mathrm{d}\tau}\delta_2\,\mathrm{d}\tau\right). \tag{7}$$

The transition probabilities between different initial and final states are also determined by the function $\delta(t)$. For example, the probability to occur in the coherent state

$$\overset{(\beta)}{\mathrm{coh}}(x_1, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[\sqrt{\frac{2m\omega}{\hbar}}\beta x_1\mathrm{e}^{-\mathrm{i}\omega t} - \frac{1}{2}\left(\frac{m\omega}{\hbar}x_1^2 + \mathrm{i}\omega t + \beta^2\mathrm{e}^{-2\mathrm{i}\omega t} + |\beta|^2\right)\right], \tag{8}$$

being initially in the ground state $\psi^{(0)}_{\rm coh}(x_1, t)$, is given by a simple formula [6]

$$\mathcal{W}_0^{\beta} = \exp\big[-|\beta + \delta(t)|^2\big]. \tag{9}$$

The transition probability from the vacuum to the nth Fock state is given by the Poisson distribution formula [1–6]

$$\mathcal{W}_0^n = \frac{|\delta(t)|^{2n}}{n!}\exp\big(-|\delta(t)|^2\big), \tag{10}$$

whereas the transition probability between an arbitrary initial and the final Fock state is expressed through the Laguerre polynomials, whose argument is again $|\delta(t)|^2$ [1,2,6],

$$\mathcal{W}_m^n = \frac{m!}{n!}|\delta(t)|^{2(n-m)}\exp\big(-|\delta(t)|^2\big)\big[L_m^{n-m}\big(|\delta(t)|^2\big)\big]^2. \tag{11}$$

In the equations above, $f(t)$ is the *classical force*, i.e. a prescribed real function of time. However, if we accept the point of view that our world has a quantum nature, then we must admit that the concept of prescribed time-dependent classical force (and, in a more general context, the concept of quantum Hamiltonians with explicitly time-dependent coefficients) is only an approximation to some more detailed description. This point is crucial, for instance, for the interpretation of time–energy uncertainty relations [7–13].

It seems reasonable to suppose that Hamiltonian (1) emerges as some kind of *mean-field approximation* to a more general Hamiltonian, which takes into account explicitly the interaction between the particle of mass m and many other particles,

$$H_{\rm tot} = \frac{\hat p_1^2}{2m} + \frac{m\omega^2}{2}\hat x_1^2 - \hat x_1\sum_{j>1}\zeta_j\hat x_j + \sum_{j>1}\left[\frac{\hat p_j^2}{2M_j} + \frac{M_j\,{}^2_j}{2}\hat x_j^2\right], \tag{12}$$

where ζ_j are coupling constants. When all coupling constants ζ_j are equal to zero exactly, mean values of each operator $\hat x_j$ and $\hat p_j$ are some harmonic functions of time,

$$\langle\hat x_j(t)\rangle = A_j\cos\big(\ {}_j t - \varphi_j\big),$$

the coefficients A_j and φ_j being determined by the initial conditions. Replacing the operators $\hat x_j$ and $\hat p_j$ by these unperturbed time-dependent mean values, one arrives, indeed, at the Hamiltonian (1), with $f(t)$ given by the Fourier series

$$f(t) = \sum_{j>1}\zeta_j A_j\cos\big(\ {}_j t - \varphi_j\big)$$

and with some time-dependent additive (thus inessential) terms.

The goal of this paper is to study to what extent this procedure of replacing the 'quantized' interaction by an equivalent classical force can be justified. Obviously, one of the necessary conditions is smallness of the coupling constants. Also, it is necessary to ensure that the motion of 'master' oscillators is not disturbed by the back-reaction of the quantum oscillator under study. At first glance, it seems reasonable to suppose that the masses of these oscillators must be large enough (since the 'macroscopic' world is usually conceived of as a world of 'heavy bodies'). However, calculations do not confirm this assumption. At least within the framework of the model considered, the masses are not important. What is important is the state of each 'master' oscillator—it must be highly excited. Moreover, this state must be close to the coherent state in order to suppress fluctuations of the 'classical force'.

We take advantage of the fact that both Hamiltonians (1) and (12) are *quadratic*, i.e. exactly solvable. The general scheme for solving the Schrödinger equation with multidimensional quadratic Hamiltonians has been given, for example, in [14, 15]. However, the explicit form of the solution appears unexpectedly cumbersome. For this reason, we confine ourselves to the simplest case of *two* coupled oscillators—the 'subordinate' ('quantum') and 'master' ('classical') ones.

The model of two coupled time-dependent harmonic oscillators has been studied by many authors, who have applied it to various problems of quantum mechanics and quantum optics. For instance, it was used to describe quantum amplifiers and converters in [16–20]. The explicit exact solutions and propagators of the Schrödinger equation, as well as solutions of the Heisenberg equations of motion, were considered in [21–27]. Squeezing, photon statistics, and entanglement in the system of two coupled oscillators were studied in [28–36]. A model of two coupled oscillators was used recently to illustrate Feynman's concept of the 'rest of the Universe' [37], to demonstrate new methods of solving the Schrödinger equation [38], and to study the exchange of nonclassical properties [39] and the exchange of quantum states without energy transfer between two modes of the electromagnetic field [40]. However, the problem of 'quantum–classical correspondence' in this system has not been analysed in detail, as far as we know (another treatment was given in [12]).

The plan of the paper is as follows. In section 2, we give the known exact solutions describing the 'forced' oscillator. In section 3, we solve exactly the problem of two coupled oscillators, obtaining the exact propagator as well as the complete wavefunction for the specific initial conditions. It appears that the possibility of reducing the problem of two coupled oscillators to the equivalent description in terms of a single forced oscillator depends on whether the frequencies of the two oscillators are different or not. Therefore, in section 4 we analyse the case of essentially different frequencies, while the specific resonance case is treated in section 5. In section 6, we consider the influence of the initial state of the 'master' oscillator and discuss the results of the paper in section 7.

2. One-dimensional oscillator under the action of a harmonic force

Confining ourselves to the single Fourier component of the time-dependent force, we assume that the function $f(t)$ in equation (1) is given by $f(t) = F\sin(\ t)$ (in principle, we could introduce an arbitrary phase in the argument of the sine function; however, such a generalization results in extremely cumbersome explicit expressions for the propagator, wavefunctions, etc, which do not provide any new physical information). It is convenient to introduce the following

notation to simplify the formulae:

$$s_1 = \sin(\omega t)/\omega, \qquad s_2 = \sin(\quad t)/\quad , \tag{13}$$

$$c_1 = \cos(\omega t), \qquad c_2 = \cos(\quad t), \tag{14}$$

$$s_{ij}^{\pm} = s_i \pm s_j\,, \qquad c_{ij}^{\pm} = c_i \pm c_j\,, \tag{15}$$

$$= \omega^2 - \quad ^2, \qquad \mathcal{Z} = \omega^2 + \quad ^2. \tag{16}$$

Then the functions $\delta_1(t)$ and $\delta_2(t)$ defined in equations (3) and (4) can be written as

$$\boldsymbol{x}$$

$$\delta_1(t) = \frac{F}{\quad}\big(1 - c_1c_2 - \omega^2 s_1 s_2\big), \tag{17}$$

$$\delta_2(t) = \frac{\omega \quad F}{\quad}\,(s_1c_2 - s_2c_1)\,, \tag{18}$$

so that the force-dependent coefficients in the propagator (5) assume the following explicit forms:

$$g(t) = \frac{F}{s_1}\,(s_1c_2 - s_2c_1)\,, \qquad \tilde{g}(t) = \frac{F s_{12}^-}{s_1}, \tag{19}$$

$$\phi(t) = \frac{F^2}{4m\quad^2}\left(\frac{2\quad{}^2 s_2^2 c_1}{s_1} - \mathcal{Z} s_2 c_2 + \quad t\right), \tag{20}$$

$$|\delta(t)|^2 = -\frac{{}^2F^2}{2m\omega\hbar\quad^2}\big[\quad s_2^2 - 2(\omega^2 s_1 s_2 + c_1c_2 - 1)\big]. \tag{21}$$

In this case, the initial coherent state $\psi^{(\beta)}_{\rm coh}(x_1, 0)$ (given by (8) with $t = 0$) evolves as follows:

$$\psi^{(\beta)}_{\rm coh}(x_1, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x_1^2 + v(t)x_1 + \varphi(t)\right). \tag{22}$$

The complex function $v(t)$ gives mean values of the position and momentum

$$v(t) = \frac{m\omega}{\hbar}\,\langle x_1(t)\rangle + \frac{\mathrm{i}}{\hbar}\,\langle p_1(t)\rangle$$
$$= \sqrt{\frac{2m\omega}{\hbar}}\beta \mathrm{e}^{-\mathrm{i}\omega t} - \frac{F}{\hbar}\left(\omega s_{12}^- + \mathrm{i}c_{12}^-\right). \tag{23}$$

The real part of the coordinate-independent factor $\varphi(t)$ is responsible for the normalization, whereas its imaginary part is necessary to satisfy the Schrödinger equation. We do not give here the explicit expressions for these functions, since they are very cumbersome and because more general expressions will appear in the following sections.

3. Two coupled oscillators

Now we turn to the system of two coupled oscillators described by the partial case of the generic Hamiltonian (12)

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{m\omega^2}{2}\hat{x}_1^2 - \zeta\hat{x}_1\hat{x}_2 + \frac{\hat{p}_2^2}{2M} + \frac{M\quad^2}{2}\hat{x}_2^2, \tag{24}$$

with a coupling constant ζ. We assume, for definiteness, that $\zeta > 0$ and $\omega \geqslant$. The Schrödinger equation with the Hamiltonian (24) can be solved exactly: for example, by introducing the normal coordinates which reduce the Hamiltonian to the sum of two uncoupled oscillator Hamiltonians. We prefer, however, to use the general scheme of treating the systems with arbitrary multidimensional quadratic Hamiltonians proposed in [14] and detailed in [15]. According to this scheme, one should represent the Hamiltonian in the matrix form

$$\hat{H} = \tfrac{1}{2}\hat{q}B\hat{q} \tag{25}$$

where $\hat{q} = \big(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N\big)$ and B is a symmetric $2N \times 2N$ matrix. Then the propagator in the coordinate x-representation can be written as

$$G(x, x', t) = [\det(-2\pi\mathrm{i}\hbar\lambda_3)]^{-1/2}$$
$$\times \exp\left[\frac{-\mathrm{i}}{2\hbar}\big(x\lambda \qquad\qquad x\right.$$

$$\xi = \quad /R, \qquad = S_2C_1 - S_1C_2, \tag{37}$$

$$A_{1,2} = \tfrac{1}{2}\left[(1 \pm \xi)S_2C_1 + (1 \mp \xi)S_1C_2\right]. \tag{38}$$

Suppose that initially the 'master' oscillator was in the *coherent state* $|\alpha\rangle$, since it is the 'most classical' quantum state. The 'subordinate' oscillator can be initially in an arbitrary state. However, in order to simplify the formulae, as far as possible, we suppose that it was also in some coherent state $|\beta\rangle$. Then the complete wavefunction of the coupled system at $t > 0$ has the Gaussian form

$$\begin{aligned}{}^{\beta\alpha}_{12}(x_1, x_2, t) = &\left(\frac{mM\omega \quad ^2}{(\pi\hbar)^2}\right)^{1/4} \\ &\times \exp\left(- \frac{m\omega}{2\hbar}x_1^2 - \frac{M}{2\hbar}x_2^2 \right. \\ &\left. + b_1x_1^2 + a_1x_2^2 + b_2x_1 + a_2x_2 + \gamma x_1x_2 + d\right).\end{aligned} \tag{39}$$

The coefficients b_1 and b_2 are given by

$$b_1 = \frac{m\nu^4}{2\hbar R^2}\left[(\omega - \quad)(2C_1C_2 + \mathcal{Z}S_1S_2) + \mathrm{i}R \quad\right], \tag{40}$$

$$b_2 = \sqrt{\frac{m\omega}{2\hbar}}\mathcal{P}(\xi; \quad)\ \beta + \frac{\zeta \quad \alpha}{R}\sqrt{\frac{2}{M\hbar}}\left(C_{12}^- + \mathrm{i}\omega S_{12}^-\right), \tag{41}$$

where

$$\mathcal{P}(\xi; \quad) = (1 - \xi)(C_1 + \mathrm{i} \quad S_1) + (1 + \xi)(C_2 + \mathrm{i} \quad S_2). \tag{42}$$

Analogous coefficients a_1 and a_2 can be obtained from b_1 and b_2 by means of the substitutions $m \leftrightarrow M$, $\omega \leftrightarrow$ and the change of coefficients $\alpha \leftrightarrow \beta$ in equation (41) (but without the change of indices in (40) and (42)), remembering that these transformations imply the replacement $\xi \to -\xi$ and $\to -$.

The coefficient γ, which is responsible for the *entanglement* of two oscillators, is proportional to the coupling constant ζ

$$\gamma = \frac{\zeta\tilde{\gamma}}{\hbar\omega_s}, \qquad \omega_s = \omega + \quad , \tag{43}$$

$$\tilde{\gamma} = 1 - \quad \left[\xi + \left(1 - \xi^2\right)\left(C_1C_2 + \tfrac{1}{2}\mathcal{Z}S_1S_2\right)\right]. \tag{44}$$

The factor is given by

$$= \left[C_1C_2 - \omega \quad S_1S_2 + \mathrm{i}\left(\chi_1S_1C_2 + \chi_2S_2C_1\right)\right]^{-1}, \tag{45}$$

with

$$\chi_{1,2} = \tfrac{1}{2}\left[(1 \pm \xi)\omega + (1 \mp \xi) \quad\right]. \tag{46}$$

If $\zeta = 0$, then $= \exp[-\mathrm{i}(\omega + \quad)t]$. The coordinate-independent coefficient d reads

$$\begin{aligned}d = &-\frac{}{2}\left[\mathrm{i}\left(\beta^2 - \alpha^2\right)(\rho_1S_1C_2 + \rho_2S_2C_1)\right. \\ &\left. -\left(\beta^2 + \alpha^2\right)(C_1C_2 + \omega \quad S_1S_2)\right] \\ &-2\mathrm{i}\nu^2\sqrt{\omega} \quad R^{-1} \quad \alpha\beta - \tfrac{1}{2}\left(|\alpha|^2 + |\beta|^2\right),\end{aligned} \tag{47}$$

where

$$\rho_{1,2} = \tfrac{1}{2}\left[(1 \pm \xi)\omega - (1 \mp \xi) \quad\right].$$

Formulae (36)–(45) are exact, they hold for any values of the frequencies ω, and ν (even in the unstable case $\nu > \sqrt{\omega}$ where the Hamiltonian (24) is not positive definite).

4. Nonresonance weak coupling

Our goal is to derive the one-dimensional wavefunction of the forced oscillator given by equations (22) and (23) from the two-dimensional complete wavefunction (39). In accordance with the reasoning given after equation (12), we assume hereafter that the coupling constant ζ is small enough. In such a case, the eigenfrequencies of the coupled system ω_1 and ω_2 are close to ω and , respectively, with the corrections proportional to ζ^2:

$$\omega_1 \approx \omega\left(1 + \frac{\nu^4}{2\omega^2}\right), \qquad \omega_2 \approx \quad \left(1 - \frac{\nu^4}{2 \quad ^2}\right). \tag{48}$$

These expressions hold provided the frequencies ω and are different. This means that the coupling frequency ν must be small not only with respect to the frequencies ω and but also with respect to their difference $|\omega - \quad|$.

Now we make the next step, assuming that the 'master' oscillator was initially in the *highly excited* ('macroscopic') coherent state, i.e. $|\alpha| \gg 1$. In accordance with the reasoning given in the introduction, we suppose that the 'classical force' in the Hamiltonian (1) originates from the mean value of the coordinate of the 'master' oscillator, $f(t) = \zeta\langle\alpha|\hat{x}_2|\alpha\rangle$. In order to obtain the dependence $f(t) = F\sin(\quad t)$, we identify the amplitude of the force with

$$F = \zeta|\alpha|\sqrt{\frac{2\hbar}{M}} \tag{49}$$

and choose the phase of the complex number α as $\alpha = \mathrm{i}|\alpha|$.

All the properties of the 'subordinate' oscillator can be obtained from the reduced density matrix

$$\rho_1^\beta(x_1, x_1', t) = \int_{-\infty}^{\infty} {}^{\beta\alpha}_{12}(x_1, x_2, t)\left[\quad {}^{\beta\alpha}_{12}(x_1', x_2, t)\right]^* \mathrm{d}x_2.$$

Calculating this integral we obtain the following reduced density matrix of the 'subordinate' oscillator:

$$\begin{aligned}\rho_1^\beta(x_1, x_1', t) = &\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[- \frac{m\omega}{2\hbar}\left(x_1^2 + x_1'^2 - \kappa x_1x_1'\right)\right. \\ &\left. + v_1(t)x_1 + v_1^*(t)x_1' + \varphi_\rho(t)\right].\end{aligned} \tag{50}$$

We have neglected the terms similar to $b_1x_1^2$ in equation (39), which are proportional to $\nu^4/\ ^2$, i.e. to the square of the coupling constant ζ. The coefficient κ is responsible for the 'impurity' of the quantum state of the first oscillator, which is the consequence of its *entanglement* with the second oscillator. However, this coefficient is very small under the assumptions made. It is proportional to the square of the 'entanglement coefficient' γ in the wavefunction (39):

$$\kappa = \frac{\hbar^2|\gamma|^2}{Mm \quad \omega} = \frac{\nu^4|\tilde{\gamma}|^2}{\omega\omega_s^2} \sim \zeta^2. \tag{51}$$

Consequently, the 'impurity' $1 - \mathrm{Tr}\,\hat{\rho}_1^2 \approx \kappa/2$ is so small that it can be neglected. Neglecting the coefficient κ we arrive at the factorized density matrix

$$\rho_1^\beta(x_1, x_1', t) \approx \quad {}^\beta_1(x_1, t)\left[\quad {}^\beta_1(x_1', t)\right]^*. \tag{52}$$

This means that, in spite of the entanglement, the state of the first oscillator indeed can be described, with a sufficient accuracy, in terms of the *wavefunction* $\overset{\beta}{1}(x_1, t)$, which has the same structure as the forced oscillator wavefunction (22). So we only have to compare the function $v(t)$ in equation (23) with the function $v_1(t)$ in equation (50). The coordinate-independent term $\varphi_\rho(t)$ can be split unambiguously into the sum $\varphi_1(t)+\varphi_1^*(t)$, if one takes into account that φ_1 must depend only on complex numbers β, α, and their absolute values, but not on β^* and α^*.

In the nonresonance case $\omega \neq$, parameter ξ defined by equations (32) and (37) is close to unity, $\xi = 1 - 2v^4/\ ^2$. Replacing frequencies ω and with ω_1 and ω_2 in equations (42) and (45), we have

$$C_2 + \mathrm{i}\ \ S_2 = \exp(\mathrm{i}\omega_2 t), \qquad = \exp[-\mathrm{i}(\omega_1 + \omega_2)\, t],$$

with an accuracy of the order of $v^4/\ ^2$. Using this approximation, we obtain for $v_1(t)$ practically the same expression as (23), with the only difference that the frequencies ω and *in the arguments of trigonometric functions* must be replaced by the shifted frequencies ω_1 and ω_2, respectively:

$$v_1(t) = \sqrt{\frac{2m\omega}{\hbar}}\beta \mathrm{e}^{-\mathrm{i}\omega_1 t} - \frac{F}{\hbar}\left(\omega S_{12}^- + \mathrm{i}C_{12}^-\right). \tag{53}$$

The coordinate-independent term reads

$$\varphi_1 = -\frac{1}{2}\beta^2 \mathrm{e}^{-2\mathrm{i}\omega_1 t} - \frac{1}{2}|\beta|^2 - \frac{\omega\ \ ^2F^2}{2\hbar m\ \ ^2}\left(S_{12}^-\right)^2 + \frac{2\ \ \omega F S_{12}^-}{\sqrt{2\hbar m\omega}}\,\beta \exp(-\mathrm{i}\omega_1 t). \tag{54}$$

Consequently, in the nonresonance case, the model of a forced oscillator is completely justified, if the phase differences $|\omega_1 - \omega|t$ and $|\omega_2 - \ \ |t$ are much less than unity. Moreover, we have a simple recipe to generalize the forced-oscillator model to take into account the effect of coupling (or 'force quantization') for any time—it is sufficient to use corrected frequencies ω_1 and ω_2 instead of given frequencies ω and . (Actually, this should be done only in the arguments of oscillating functions, since corrections of the order of ζ^2 in the amplitudes of such functions are always unimportant.)

The same result, namely, the one-dimensional wavefunction (22), can be obtained from the two-dimensional wavefunction (39) with the aid of the 'mean-field approximation', if one simply replaces the variable x_2 in the right-hand side of (39) by its mean value in the 'free' coherent state

$$\langle x_2\rangle = |\alpha|\sqrt{\frac{2\hbar}{M}}\quad s_2,$$

neglecting the difference between the initial frequencies ω, and the shifted frequencies ω_1, ω_2. One more possibility is to multiply the total wavefunction $\overset{\beta\alpha}{12}(x_1, x_2, t)$ by the (complex conjugated) unperturbed time-dependent wavefunction of the 'master' oscillator $\left[\ \overset{\alpha}{\mathrm{coh}}(x_2, t)\right]^*$ and to integrate the product over the coordinate x_2. Then the same function (22) will appear, provided one neglects again differences between the frequencies ω, and ω_1, ω_2. Strictly speaking, both these 'brute force' methods result in an unnormalized wavefunction of the first oscillator, but there is no problem in restoring the normalization, if the total dependence on the coordinate x_1 is known.

The reduced density matrix (or the effective wavefunction) of the 'master' oscillator can be calculated in the same way, and the formulae are similar to those given above, with changing masses $m \leftrightarrow M$, frequencies $\omega \leftrightarrow$, and indices $1 \leftrightarrow 2$. Consequently, mean values of the coordinate and momentum of the 'master' oscillator are shifted from their 'free' values due to the terms caused by the 'back-reaction force' with the amplitude

$$F_{\mathrm{BR}} = \zeta|\beta|\sqrt{\frac{2\hbar}{m\omega}}.$$

Now we are able to find some relations under which the equivalence between the forced-oscillator model and the 'mean-field approximation' in the coupled-oscillator model can be established. Note that the natural scale for the coordinate change of quantum oscillator in the coherent state is the mean-squared vacuum fluctuation $\sqrt{\hbar/m\omega}$. We suppose that the influence of the 'master' oscillator on the 'subordinate' oscillator is strong enough to change its mean position at least by this value. This is equivalent to the requirement that the absolute value of the second term (proportional to F) in the right-hand side of equation (53) is of the order of $\sqrt{m\omega/\hbar}$ (or greater). Thus, we arrive at the condition $|\alpha|v^2/(\omega\sqrt{\omega\ \ }) > 1$, where v is the coupling frequency (31), which is assumed to be much smaller than $\sqrt{\omega\ \ }$ and ω (recall that we suppose that $\omega >$). Therefore, the 'master' oscillator must be highly excited, $|\alpha| \gg 1$, in order to influence the motion of the 'subordinate' oscillator.

On the other hand, the back-reaction from the 'subordinate' oscillator on the 'master' oscillator must be weak, resulting in small additional displacements compared with that in the coherent state $|\alpha\rangle$. This means that the second term in the expression obtained from (53) by means of replacement $m \leftrightarrow M$, $\omega \leftrightarrow$, $1 \leftrightarrow 2$, $\alpha \leftrightarrow \beta$ and $F \leftrightarrow F_{\mathrm{BR}}$ must be much smaller than the first one (in absolute value). In this way, we find the second condition $|\beta|v^2/(\omega\sqrt{\omega\ \ }) \ll |\alpha|$, which means that the 'subordinate' oscillator should not be 'too excited' at the initial moment. Note that the values of masses of both oscillators turn out to be inessential, since all the inequalities are expressed only in terms of the frequencies ω, , v and excitation parameters $|\beta|$, $|\alpha|$. In other words, the 'classical' oscillator need not be 'heavy' but its coherent state must be highly excited.

As long as the evolution of the 'subordinate' oscillator can be described in the weak coupling limit in terms of the wavefunction $\overset{(\beta)}{1}(x_1, t)$, the scalar product of this function with the free coherent state wavefunction $\overset{(\mu)}{\mathrm{coh}}(x_1, t)$ (8) gives the transition amplitude $\mathcal{A}_\beta^\mu$ from the initial coherent state $|\beta, 0\rangle$ to an arbitrary free coherent state $|\mu, t\rangle$. Clearly, this amplitude can be expressed as some Gaussian exponential of complex variables β and μ [6, 41, 42]:

$$\mathcal{A}_\beta^\mu = \mathcal{A}_0^0 \exp\left[-\tfrac{1}{2}\left(|\beta|^2 + |\mu|^2 + R_{11}\mu^{*2} + R_{22}\beta^2\right) - R_{12}\beta\mu^* + \mu^* z_1 + \beta z_2\right], \tag{55}$$

where the time-dependent complex coefficients R_{ij} form a symmetric 2×2 matrix $\mathcal{R}$, and complex coefficients z_1, z_2 (also time-dependent) can be considered as components of a two-dimensional vector z. $\mathcal{A}_0^0(t)$ is the 'survival amplitude'

of the oscillator's ground state. The function $\mathcal{A}_\beta^\mu$ is the generating function for the transition amplitudes between the energy eigenstates of the oscillator. On the other hand, the Gaussian exponential is the generating function of two-dimensional Hermite polynomials $H_{mn}^{\{\mathcal{R}\}}(y_1, y_2)$ defined by matrix $\mathcal{R}$. Therefore [6, 41–43]

$$\mathcal{A}_n^m = \mathcal{A}_0^0 \frac{H_{mn}^{\{\mathcal{R}\}}(y_1, y_2)}{\sqrt{n!m!}}, \qquad \boldsymbol{y} = \mathcal{R}^{-1}\boldsymbol{z}. \tag{56}$$

In the case involved, only off-diagonal elements of symmetrical matrix $\mathcal{R}$ are significant, $\mathcal{R} = -\sigma_x + \delta\mathcal{R}$, where σ_x is the Pauli matrix, and all elements of the correction matrix $\delta\mathcal{R}$ are small (they are of the order of ν^4/ω). With the same accuracy, $y_1 = \delta(t)$ and $y_2 = -\delta^*(t)$. In this specific case, two-dimensional Hermite polynomials are reduced to associated Laguerre polynomials according to the formula [15, 43]

$$H_{mn}^{\{\tau\sigma_x\}}(y_1, y_2) = n!\tau^m(-1)^n y_2^{m-n} L_n^{m-n}(\tau y_1 y_2), \tag{57}$$

where the inequality $m \geqslant n$ is assumed (if $n > m$, one should interchange suffices m and n in the right-hand side, and replace $y_2 \leftrightarrow y_1$). Then equations (56) and (57) give the same transition probability (11) as the forced-oscillator model.

To illustrate the results obtained, we have calculated mean values of the coordinates and momenta using the exact wavefunction (39) and compared them with the prediction of the forced-oscillator model. These mean values are determined by elements of the evolution matrix (t) (given in the appendix) according to the vector relation

$$\langle\hat{Q}(t)\rangle = {}^{-1}(t)\langle\hat{Q}(0)\rangle, \tag{58}$$

where the four-dimensional vector $\hat{Q}$ has the components $(\hat{p}_1, \hat{p}_2, \hat{x}_1, \hat{x}_2)$. Since matrix (t) is symplectic, the inverse matrix ${}^{-1}$ can be expressed in terms of the transposed matrix $\tilde{}$ as

$${}^{-1} = - \quad \tilde{} \quad , \tag{59}$$

where the antisymmetric matrix was defined in (27).

Figure 1 shows 'mean trajectories' in the phase space for two oscillators. Coefficients of the Hamiltonian (24) are as follows: $= 0.1\,\omega$, $M = 100\,m$, $\alpha = 32\,\mathrm{i}$, $\beta = 1$ and $\zeta = 0.12$. We use dimensionless values, assuming formally $\hbar = \omega = m = 1$. In order to put two trajectories in one plot, we use different scales for two particles—for the 'master' oscillator, we have plotted $p_2/100$ and $x_2/5$.

The heavy curve represents the trajectory of the 'master' oscillator, while the light curve represents the motion of the 'subordinate' oscillator. We see that, despite the 'coupling frequency' ν having the same order of magnitude as the 'master' oscillator eigenfrequency , the motion of the 'master' particle is practically unperturbed, and its trajectory is very close to the ellipse defined by the equations

$$\langle p_2(t)\rangle = \sqrt{2M \quad \hbar}\,|\alpha|\cos(\quad t),$$

$$\langle x_2(t)\rangle = \sqrt{\frac{2\hbar}{M}}\,|\alpha|\sin(\quad t).$$

At the same time, the motion of the 'subordinate'

2 ω

The simplest way to calculate the evolution of the second-order moments $\langle \hat{Q}_i \hat{Q}_j \rangle$ is to use the matrix formula

$$\mathcal{M}(t) = \quad ^{-1}(t)\mathcal{M}(0)\tilde{}^{-1}(t) \tag{60}$$

for the matrix $\mathcal{M}$ constructed from these statistical moments. The explicit expression for the energy of the first oscillator in the nonresonance case is

$$\langle E_1(t) \rangle = \hbar\omega \left(|\beta|^2 + \frac{1}{2} \right) + \frac{\omega \quad F}{} \sqrt{\frac{2\hbar}{m\omega}}$$
$$\times \mathrm{Im} \left\{ \beta \left[(C_2 + \mathrm{i}\omega S \right. \right.$$

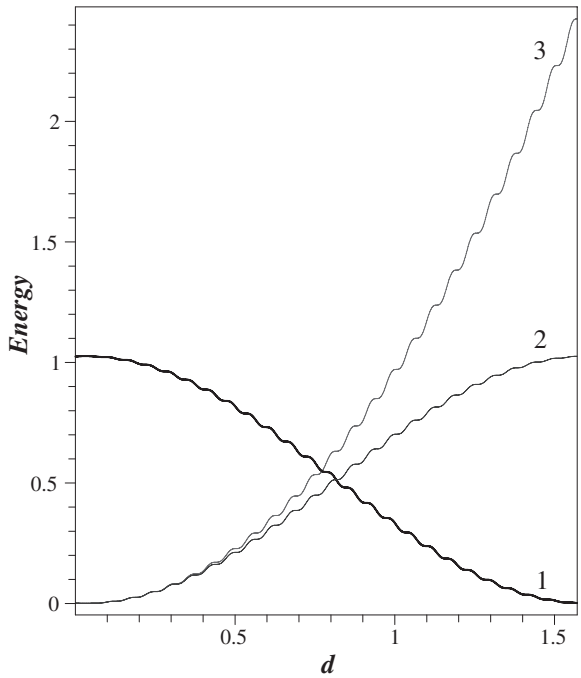

Figure 3. Energy of the 'master' (1), 'subordinate' (2), and 'forced' (3) oscillators under the resonance conditions versus $d = \omega_d t/2$.

Equations (67) and (68) show that the 'subordinate' and 'master' oscillators exchange their energies with the course of time. If the ratio

$$k = \frac{s}{d} = \left(\frac{\omega}{\nu}\right)^2 + \sqrt{\left(\frac{\omega}{\nu}\right)^4 - 1} \approx 2\left(\frac{\omega}{\nu}\right)^2$$

is a rational number p/q, then the process is strictly periodic with period $T_k = 2\pi q/\ _d$; otherwise, the process of energy exchange is 'almost periodic'. In figure 3, we have plotted dependences of dimensionless energies of the 'subordinate' and 'master' oscillators versus the dimensionless parameter $d = \ _d t/2$ in the case where $k = 50$, choosing $M = 100\,m$, $m = \omega = 1$ (therefore $\zeta \simeq 0.39$), $\alpha = 32\,\mathrm{i}$, and $\beta = 1$ (since the dependences are periodic, only the first half-period is shown). The light curve 3 shows an unlimited growth of the energy of the 'forced' oscillator under the resonance force with the amplitude given by (49) with the same parameters. Until $t \ll 1/\ _d$, this curve coincides with that representing the mean energy of the 'coupled' oscillator, but for larger values of time the two models give completely different results.

Note that at the time moment when $\ _d t/2 = \pi/2$, we have not only the exchange of energies, but the complete exchange of quantum states between two oscillators. This is clearly seen from equation (65), which shows that the state of the first oscillator, which was initially the coherent state $|\beta\rangle$, is transformed to another coherent state, whose amplitude $|\alpha|$ exactly equals the initial amplitude of the state of the 'master' oscillator. Other examples of the exchange of quantum states between interacting modes of the electromagnetic field (or equivalent oscillators) have been studied recently: for example, in [39, 40].

6. Influence of the initial state

Now let us consider the situation where the initial state of the 'master' oscillator is different from the coherent state. Suppose, first, that it is not pure, but a quantum mixture: for example, the displaced thermal state with the same mean values of the coordinate and momentum as in the coherent state $|\alpha\rangle$, but with variances corresponding to the thermal state with temperature T (this is the 'most classical' mixed state [44]). Since the density matrix of this state is Gaussian, the density matrix of the total system will remain Gaussian for all moments of time, as will the reduced density matrix of the 'subordinate' oscillator (we still assume that initially it was in the coherent state). But it is well known that properties of Gaussian states are determined completely by mean values of the coordinates in the phase space and the variances; moreover, these quantities evolve in time independently of each other. Therefore, it is sufficient to calculate the mean values using equation (58) and the variance matrix using equation (60) and appropriate initial conditions.

Obviously, the evolution of the first-order mean values is the same as in the case of a pure state, for the same initial conditions. Moreover, one can verify that *in the nonresonance case* the variance matrix of the 'subordinate' oscillator does not change with time, within small corrections of the order of $(\nu^4/\ ^2)(\omega/\ )(k_{\mathrm{B}}T/\hbar\ )$ (we consider for simplicity the high-temperature limit $k_{\mathrm{B}}T \gg \hbar\omega$, where k_{B} is Boltzmann's constant). Consequently, the 'subordinate' oscillator remains in the pure coherent state even when coupled to the highly mixed 'master' oscillator, and the forced-oscillator model as well as the 'mean-field approximation' still hold.

The situation is quite different in the resonance case where elements of the variance matrix of the 'subordinate' oscillator vary in time as follows (in the high-temperature limit, for simplicity):

$$\mathcal{M}_{11} \simeq \frac{1}{2}m\hbar\omega\cos^2\frac{\ _d t}{2} + mk_{\mathrm{B}}T\left(\sin^2\frac{\ _d t}{2} + \frac{\nu^2}{4\omega^2}\sin\ \ _s t\sin\ \ _d t\right), \tag{70}$$

$$\mathcal{M}_{22} \simeq \frac{\hbar}{2m\omega}\cos^2\frac{\ _d t}{2} + \frac{k_{\mathrm{B}}T}{m\omega^2}\left(\sin^2\frac{\ _d t}{2} - \frac{\nu^2}{4\omega^2}\sin\ \ _s t\sin\ \ _d t\right), \tag{71}$$

$$\mathcal{M}_{12} = \mathcal{M}_{21} \simeq -\frac{k_{\mathrm{B}}T\nu^2}{2\omega^3}\sin\ \ _d t\sin^2\frac{\ _s t}{2}, \tag{72}$$

where

$$\mathcal{M}_{\mu\nu} = \tfrac{1}{2}\langle\hat{q}_\mu\hat{q}_\nu + \hat{q}_\nu\hat{q}_\mu\rangle - \langle\hat{q}_\mu\rangle\langle\hat{q}_\nu\rangle,$$
$$\hat{q}_1 = \hat{p}_1, \qquad \hat{q}_2 = \hat{x}_1.$$

The variances preserve their initial values (i.e. the coherent state remains coherent, although with a time-dependent amplitude), as far as $\ _d t \ll 1$. When $\ _d t = \pi$ the 'subordinate' oscillator occurs in the (displaced) thermal state with temperature T (whereas the state of the 'master' oscillator is completely 'purified', since it becomes the coherent state $|\beta\rangle$). This is another example of the 'quantum-state exchange' between two systems under the resonance condition.

Now let us suppose that initially the 'master' oscillator was in a *non-Gaussian* pure state: for example, in a combination

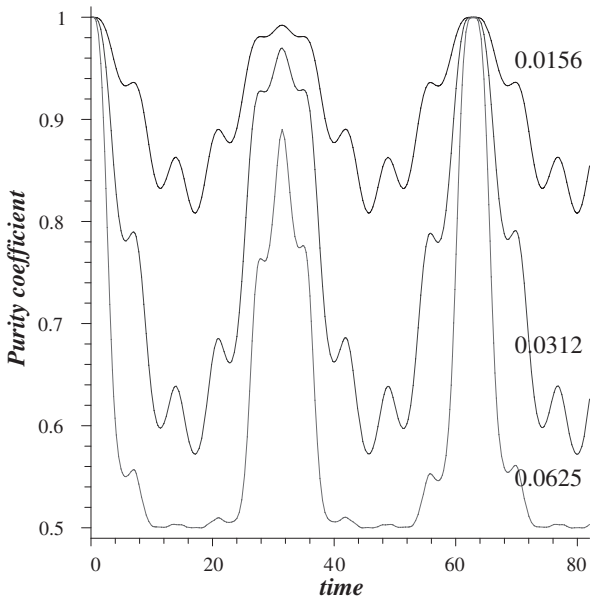

Figure 4. The purity coefficient of the 'subordinate' oscillator $\mu \equiv \mathrm{Tr}\,\hat{\rho}_1^2$ versus dimensionless time, when the 'master' oscillator was initially in the superposition of coherent states $|0\rangle$ and $|\alpha\rangle$ with $|\alpha| = 128$ for $\omega = 10$, $\quad = 1$, and different values of the ratio ν^2/ω as indicated in the plot.

of quasi-orthogonal coherent states $|0\rangle$ and $|\alpha\rangle$, with $|\alpha| \gg 1$ (assuming that the initial state of the 'subordinate' oscillator is still the coherent state $|\beta\rangle$). In this case, the reduced density matrix of the 'subordinate' oscillator is a sum of four Gaussians, each having the form (50), with exactly the same coefficients at x_1^2, $x_1'^2$, and $x_1 x_1'$, but with different coefficients v_1, v_1^* (they are not necessarily complex conjugated now), and φ_ρ. Consequently, the equivalence between the forced-oscillator model (which yields only one time-dependent Gaussian wavefunction for the initial coherent state) and the coupled-oscillator model is evidently broken. This is confirmed by the calculation of the *purity* $\mu \equiv \mathrm{Tr}\,\hat{\rho}_1^2$ of the state of the 'subordinate' oscillator. Taking $\alpha = \mathrm{i}|\alpha|$ and neglecting small corrections of the order of ζ^2 and $\exp(-|\alpha|^2)$, one can obtain after some algebra the following expression:

$$\mu(t) = \frac{1}{2}\left\{1 + \exp\left(-\frac{|\alpha|^2\nu^4}{\omega^{\quad 2}}\Big[{}^{\quad 2}\big(\cos\omega_1 t - \cos\omega_2 t\big)^2 + \big(\quad \sin\omega_1 t - \omega\sin\omega_2 t\big)^2\Big]\right)\right\}. \tag{73}$$

In the forced-oscillator model, we have $\mu \equiv 1$. Typical dependences $\mu(t)$ for different values of the ratio ν^2/ω are shown in figure 4. Function $\mu(t)$ is periodic, provided the ratio $r \equiv \omega_1/\omega_2$ is a rational number. If $r = p/q > 1$, then the period of $\mu(t)$ is $\mathcal{T} = 2\pi q/\omega_2$, if integers q and p have different parities, whereas $\mathcal{T} = \pi q/\omega_2$, if both q and p are odd numbers.

The impossibility of reducing the coupled-oscillator model to the forced-oscillator model is explained by strong fluctuations of the coordinate of the 'master' oscillator (and, consequently, the corresponding 'force' ζx_2) in the superposition state involved. Indeed, neglecting small corrections we have for $\mathrm{Re}\,\alpha = 0$

$$\langle x_2\rangle \approx \sqrt{\frac{\hbar}{2M}}\,|\alpha|\sin(\quad t),$$

$$\mathcal{M}_{x_2x_2}(t) \approx \frac{\hbar}{2M}\big(1 + |\alpha|^2\sin^2\quad t\big).$$

Consequently, the fluctuations of the coordinate x_2 always exceed its mean value, and the model of well defined time-dependent classical force cannot work in this case.

7. Conclusions

We have shown explicitly under which conditions the model of two weakly coupled quantum oscillators gives the same results as a simpler model of a single quantum oscillator driven by a classical force. The 'master' oscillator must be in a highly excited quantum state with small fluctuations of its coordinate (in the case of the coupling via coordinates considered in the paper). The best state is coherent state $|\alpha\rangle$ with $|\alpha| \gg 1$, although other Gaussian states are also admitted: in particular, displaced thermal states. Under the condition $\nu^4\omega k_\mathrm{B}T/[(\omega^2 - \quad^2)^2\hbar\quad^2] \ll 1$, the 'subordinate' oscillator remains in the pure quantum state (if it was in such a state initially) even for highly mixed states (such as high-temperature displaced thermal states) of the 'master' oscillator (where ω and $\quad$ are frequencies of the 'subordinate' and 'master' oscillators, respectively, and ν is the 'coupling frequency'). In the resonance case $\omega = \quad$, two models are equivalent only under the condition $\nu^2 t/\omega \ll 1$: i.e. for a limited initial interval of time. Moreover, in the resonance case, two oscillators exchange their initial quantum states at instants of time $t_k = (2k+1)\pi\omega/\nu^2$, $k = 0, 1, \ldots$. For essentially non-Gaussian states of the 'master' oscillator, such as, for instance, superpositions of coherent states with significantly different parameters α_1 and α_2, no equivalence between the two models exists, due to strong fluctuations of the 'effective force' acting on the 'subordinate' oscillator. The ratio of the masses of the two oscillators has no importance, which means that the 'classical' oscillator is not obliged to be 'heavy' (in the context of the problem discussed), but its quantum state must be highly excited. By contrast, the 'subordinate' oscillator should not be 'too excited', in order not to cause the change of the state of the 'master' oscillator, which would result in large fluctuations of the 'equivalent force'. In particular, in the case of initial coherent states $|\alpha\rangle$ and $|\beta\rangle$ of the 'master' and 'subordinate' oscillators, respectively, the inequality $|\beta|\nu^2/(\omega\sqrt{\omega\quad}) \ll |\alpha|$ must hold.

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Appendix. Elements of matrix $\Lambda(t)$

Here, we give explicit exact expressions for all elements of the 4×4 matrix (t), which determine the evolution of the system of two coupled oscillators described by the Hamiltonian (24). This matrix satisfies equation (27), being divided into 2×2

blocks according to (28). For convenience, we put the number of each matrix λ_j in the superscript, i.e. $\lambda_{kl}^{(j)}$ means the element of the kth row and lth column of matrix λ_j:

$$\lambda_{11}^{(1)} = \tfrac{1}{2}\left(C_{12}^{+} + \xi C_{12}^{-}\right), \qquad \lambda_{12}^{(1)} = -\frac{\zeta C_{12}^{-}}{MR},$$

$$\lambda_{21}^{(1)} = -\frac{\zeta C_{12}^{-}}{mR}, \qquad \lambda_{22}^{(1)} = \tfrac{1}{2}\left(C_{12}^{+} - \xi C_{12}^{-}\right),$$

$$\lambda_{11}^{(2)} = \frac{m}{2}\left(F_{+} + \xi F_{-}\right), \qquad \lambda_{12}^{(2)} = \lambda_{21}^{(2)} = -\frac{\zeta}{R}F_{-},$$

$$\lambda_{22}^{(2)} = \frac{M}{2}\left(F_{+} - \xi F_{-}\right), \qquad F_{\pm} = \omega_1^2 S_1 \pm \omega_2^2 S_2,$$

$$\lambda_{11}^{(3)} = -\frac{\left(S_{12}^{+} + \xi S_{12}^{-}\right)}{2m}, \qquad \lambda_{12}^{(3)} = \frac{\zeta S_{12}^{-}}{mMR},$$

$$\lambda_{21}^{(3)} = \frac{\zeta S_{12}^{-}}{mMR}, \qquad \lambda_{22}^{(3)} = -\frac{\left(S_{12}^{+} - \xi S_{12}^{-}\right)}{2M},$$

$$\lambda_{11}^{(4)} = \tfrac{1}{2}\left(C_{12}^{+} + \xi C_{12}^{-}\right), \qquad \lambda_{12}^{(4)} = -\frac{\zeta C_{12}^{-}}{mR},$$

$$\lambda_{21}^{(4)} = -\frac{\zeta C_{12}^{-}}{MR}, \qquad \lambda_{22}^{(4)} = \tfrac{1}{2}\left(C_{12}^{+} - \xi C_{12}^{-}\right).$$

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