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Quantum state exchange between coupled modes

A S M de Castro[1,2], V V Dodonov[1,3] and S S Mizrahi[1]

[1] Departamento de Física, Universidade Federal de São Carlos, Via Washington Luiz km 235, 13565-905 São Carlos, SP, Brazil
[2] Departamento de Física, Universidade Estadual de Ponta Grossa, Campus de Uvaranas, Avenida Carlos Cavalcanti 4748, 84030-900 Ponta Grossa, PR, Brazil

E-mail: semaca@uepg.br, vdodonov@df.ufscar.br and salomon@df.ufscar.br

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Abstract
We study the problem of quantum state exchange between two coupled modes of the electromagnetic field modelled by quantum oscillators. Analysing the structure of propagators of the Schrödinger equation with the most general weak bilinear resonance coupling, we find the conditions of the exchange in both 'narrow' and 'wide' senses.

Keywords: Coupled modes, quantum state exchange, propagators, resonance, squeezed states, Fock states

1. Introduction

Recently, the problem of so-called 'quantum teleportation', or a transfer of the state of some quantum system to another quantum system, has attracted the attention of many authors. There exist various schemes [1–8], some of which have been realized already [9, 10]. The aim of our paper is to study the possibilities of the quantum state exchange between two coupled modes of the electromagnetic field modelled by coupled quantum oscillators.

It is a well known effect of classical mechanics that two weakly coupled identical oscillators periodically exchange their energies. For example, if initially one oscillator was at rest and another was excited, then eventually the amplitude of oscillations of the second oscillator goes to zero, while the first one acquires all the initial energy (in the absence of damping). We address the following problem: is it possible to cause the oscillators to exchange not only energies, but also their quantum states? Such an exchange can be interpreted also as an ideal information transfer [11]. Therefore, the results of our study can be applied to the problems of quantum communications and entanglement.

In some special cases (e.g., for particular initial states, such as squeezed states or coherent states and their 'cat' superpositions, or for some specific couplings between the modes) this problem was studied recently in [11–16]. Our goal is to consider *arbitrary* initial states and the most general (time-independent and time-dependent) *bilinear* coupling.

[3] On leave from: Lebedev Physics Institute and Moscow Institute of Physics and Technology, Moscow, Russia.

2. General considerations

2.1. Two kinds of state exchange

We consider a quantum system described in terms of the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2) + \gamma_1 p_1 p_2 + \gamma_2 p_1 x_2 + \gamma_3 x_1 p_2 + \gamma_4 x_1 x_2. \quad (1)$$

The masses of both the oscillators can always be made equal to unity by means of some scaling transformations. Actually, the masses are not important for our analysis (this is obvious, if x_1, x_2, p_1, p_2 are quadrature components of two coupled electromagnetic modes). There exist several different techniques to treat the systems like (1). For example, in [11] the method of characteristic functions was used, because it works well for the coherent states and their superpositions, which were the subject of interest in that paper. However, to treat the most general case, it seems better to study the structure of propagators in the coordinate representation.

Dealing with the problem of the quantum state exchange, we assume that the initial wavefunction of the total system is factorized,

$$\psi(x_1, x_2; 0) = \psi_1(x_1; 0)\,\psi_2(x_2; 0). \quad (2)$$

Then the wavefunction at any moment of time $t > 0$ can be written as

$$\psi(x_1, x_2; t) = \int G(x_1, x_2; x_1', x_2'; t)\psi(x_1', x_2'; 0)\,\mathrm{d}x_1'\,\mathrm{d}x_2', \tag{3}$$

where $G(x_1, x_2; x_1', x_2'; t)$ is the propagator of the Schrödinger equation with Hamiltonian (1), i.e. the solution to the time-dependent Schrödinger equation, dependent on two parameters, x_1' and x_2' in such a way that at the initial moment $t = 0$ one has

$$G(x_1, x_2; x_1', x_2'; 0) = \delta(x_1 - x_1')\delta(x_2 - x_2').$$

In a narrow sense, the state exchange means that at some instant of time t_* the total wavefunction is factorized again into a product of the initial terms, but with exchanged arguments:

$$\psi(x_1, x_2; t_*) = \mathcal{N}\psi_1(s_1x_2; 0)\psi_2(s_2x_1; 0). \tag{4}$$

Writing the exchange condition in the form (4) we admit a possibility that the variables x_1 and x_2 could be scaled by factors $s_{2,1}$ and some phase factor $\mathcal{N}$ could appear. To ensure (4), the propagator at the instant t_* must turn into a product of two delta-functions:

$$G(x_1, x_2; x_1', x_2'; t_*) = \mathcal{N}\delta(s_1x_2 - x_1')\delta(s_2x_1 - x_2'). \tag{5}$$

Having reached this point, we must discuss the following important question: what do we really have in mind when saying that two quantum states are 'the same'? Consider, for example, the initial energy eigenstate $\psi_E(x; 0)$. It evolves in time as $\psi_E(x; t) = \exp(-\mathrm{i}Et/\hbar)\psi_E(x; 0)$, i.e. without changing its form $|\psi(x)|^2$. Thus it is clear that functions $\psi_E(x; 0)$ and $\psi_E(x; t)$ describe the same quantum state. However, let us consider the coherent state of a field mode or a material harmonic oscillator with $m = \omega = \hbar = 1$. The free evolution of the coherent state is well known: $\psi(x; t; \alpha) = \exp(-\mathrm{i}t/2)\psi(x; 0; \alpha\mathrm{e}^{-\mathrm{i}t})$, and now $|\psi(x; t; \alpha)|^2 \neq |\psi(x; 0; \alpha)|^2$. Or consider an initial squeezed state $\psi_s(x; t)$ with a small coordinate dispersion. In a quarter of a period this dispersion attains maximal value, greater than that in a coherent state, while the momentum dispersion becomes small. Is the state $\psi_s(x; 0)$ the same as the state $\psi_s(x; \pi/2)$?

Certainly, the answers to these questions depend on the concrete physical problem, in particular, on the meaning of the generalized coordinate x. For a material particle variable x means a real space coordinate and $|\psi(x)|^2$ has a real physical meaning as the probability density in the coordinate space, which can be measured. In such a case, the states squeezed with respect to x seem quite different from the states squeezed with respect to p. Or, for example, in the case of a free particle the initial well localized Gaussian state seems quite different from the (also Gaussian) state with large coordinate dispersion and nonzero correlation between the coordinate and momentum, which arises in the process of a free evolution. In these cases, the only reasonable definition of the state exchange seems to be (4). We shall call it 'state exchange in a narrow sense' or 'state exchange of the first kind'.

A more flexible interpretation can be used for the electromagnetic field modes, when x and p have no direct physical meaning, being simply suitable 'quadrature components' of the field, which can be always rotated in the phase plane (x, p). In this case the states $|\alpha\rangle$ and $|\alpha\mathrm{e}^{-\mathrm{i}t}\rangle$ differ only in the positions of the centre of the distribution in the phase plane, so that one of them is nothing but the other taken at a shifted instance of time. The same can be said about squeezed states: during free field evolution the distribution simply rotates in the phase plane in such a way that variances and mean values periodically change in time, but the invariant squeezing parameter [17, 18] remains the same. Therefore, in certain situations one has reasons to believe that all vectors in the Hilbert space of the one-mode system, which have the form $\hat{U}_f(t)|\psi; 0\rangle$ (where $\hat{U}_f(t)$ is a unitary operator describing the free evolution of the field mode) belong to 'the same' quantum state (giving its representations for different instants of time). This is an evident generalization of the set of equivalent energy eigenstates $\exp(-\mathrm{i}Et/\hbar)\psi_E(x; 0)$. If such a point of view is accepted, then the following condition of the 'state exchange in a wide sense' or 'state exchange of the second kind' arises:

$$\psi(x_1, x_2; t_*) = \mathcal{N}\psi_1(s_1x_2; t_1)\psi_2(s_2x_1; t_2). \tag{6}$$

Note that time parameters t_1 and t_2 can be different. In the propagator language condition (6) reads

$$G(x_1, x_2; x_1', x_2'; t_*) = \mathcal{N}G_1^{(f)}(s_1x_2; x_1'; t_1)G_2^{(f)}(s_2x_1; x_2'; t_2) \tag{7}$$

where $G_k^{(f)}(x; x'; t)$ is the free propagator of the isolated kth mode.

In the following sections we show that the state exchange between weakly coupled modes (oscillators) is always possible in the wide sense (for arbitrary initial states) for any coupling coefficients. In contrast, the state exchange of the first kind can happen only for specific choices of coupling constants.

2.2. Propagator for coupled oscillators

The propagator can be calculated by means of many different methods. The most direct way is to use the general expression for the propagator of the Schrödinger equation with an arbitrary quadratic multidimensional time-dependent Hamiltonian [19, 20], and to adjust it to the special case under study.

A generic (homogeneous, for the sake of simplicity) quadratic Hamiltonian can be written as

$$H = \tfrac{1}{2}(\boldsymbol{p}, \boldsymbol{q})\mathcal{B}\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{pmatrix},$$

where $\boldsymbol{q}$ is the N-dimensional coordinate vector and $\boldsymbol{p}$ is the canonically conjugated momentum vector whose operator components obey the commutation relations (we assume $\hbar = 1$) $[\hat{q}_j, \hat{p}_k] = \mathrm{i}\delta_{jk}$, $j, k = 1, \ldots, N$. $\mathcal{B}(t)$ is a quite arbitrary symmetrical (but not necessarily real) $2N \times 2N$ matrix

$$\mathcal{B} = \begin{Vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Vmatrix}$$

consisting of four $N \times N$ blocks satisfying the conditions $b_1 = \tilde{b}_1$, $b_4 = \tilde{b}_4$ and $b_2 = \tilde{b}_3$ (tilde means matrix transposition). The explicit form of the propagator in the

coordinate representation found in [19, 20] reads

$$G(\boldsymbol{x}_2;\boldsymbol{x}_1;t) = [\det(-2\pi \mathrm{i}\lambda_3)]^{-1/2} \times \exp\left\{-\frac{\mathrm{i}}{2}[\boldsymbol{x}_2\lambda_3^{-1}\lambda_4\boldsymbol{x}_2 - 2\boldsymbol{x}_2\lambda_3^{-1}\boldsymbol{x}_1 + \boldsymbol{x}_1\lambda_1\lambda_3^{-1}\boldsymbol{x}_1]\right\} \tag{8}$$

where $N \times N$ matrices $\lambda_j(t)$ satisfy the following system of linear differential equations and initial conditions (here E_N means the $N \times N$ unit matrix):

$$\dot{\lambda}_1 = \lambda_1 b_3 - \lambda_2 b_1, \qquad \lambda_1(0) = E_N, \tag{9}$$

$$\dot{\lambda}_2 = \lambda_1 b_4 - \lambda_2 b_2, \qquad \lambda_2(0) = 0, \tag{10}$$

$$\dot{\lambda}_3 = \lambda_3 b_3 - \lambda_4 b_1, \qquad \lambda_3(0) = 0, \tag{11}$$

$$\dot{\lambda}_4 = \lambda_3 b_4 - \lambda_4 b_2, \qquad \lambda_4(0) = E_N. \tag{12}$$

In the case of Hamiltonian (1) the matrices b_j read

$$b_1 = \begin{Vmatrix} 1 & \gamma_1 \\ \gamma_1 & 1 \end{Vmatrix}, \qquad b_4 = \begin{Vmatrix} \omega_1^2 & \gamma_4 \\ \gamma_4 & \omega_2^2 \end{Vmatrix}, \qquad b_2 = \begin{Vmatrix} 0 & \gamma_2 \\ \gamma_3 & 0 \end{Vmatrix}.$$

Consequently, from the technical point of view, the problem is reduced to the search for the explicit form of the λ-matrices and the analysis of the structure of emerging quadratic forms in the argument of the propagator.

3. Time-independent coupling

3.1. Calculating λ-matrices

It is almost evident that the state exchange can occur only under the resonance conditions. Explicitly, this was demonstrated, for example, in [21] in the special case of the coordinate coupling. As a matter of fact, the model of two coupled time-dependent harmonic oscillators has been studied by many authors, who applied it to various problems of quantum mechanics and quantum optics. For instance, it was used to describe quantum amplifiers and converters in [22–26]. Explicit exact solutions and propagators of the Schrödinger equation, as well as solutions of the Heisenberg equations of motion, were considered and applied to different problems in [27–35]. Squeezing, photon statistics and entanglement in the system of two coupled oscillators were studied in [36–45]. However, in all of those papers, as a rule, only specific couplings (mainly, via coordinates) were considered, while the problem of quantum state exchange was not touched upon at all.

In this section we analyse the structure of the quantum propagator in the generic time-independent weak-coupling case. Then the resonance condition is $\omega_1 = \omega_2$. To simplify formulae, we assume that $\omega_1 = \omega_2 = 1$, $m_1 = m_2 = 1$ and $\hbar = 1$. Two first-order equations (11) and (12) can be reduced to a single second-order equation for matrix λ_3. Assuming that all coupling constants are small, $|\gamma_k| \ll 1$, $k = 1, 2, 3, 4$, and neglecting small terms of the order of $\gamma_j\gamma_k$, we obtain the equation

$$\dddot{\lambda}_3 + \dot{\lambda}_3 \begin{Vmatrix} 0 & \nu \\ -\nu & 0 \end{Vmatrix} + \lambda_3 \begin{Vmatrix} 1 & \mu \\ \mu & 1 \end{Vmatrix} = 0, \tag{13}$$

where

$$\mu = \gamma_1 + \gamma_4, \qquad \nu = \gamma_2 - \gamma_3. \tag{14}$$

The initial conditions to equation (13) are

$$\lambda_3(0) = 0, \qquad \dot{\lambda}_3(0) = -b_1. \tag{15}$$

Matrix λ_1 satisfies the same equation (13); the difference is only in the initial conditions

$$\lambda_1(0) = E_2, \qquad \dot{\lambda}_1(0) = b_3.$$

Equation (13) has constant coefficients, so it can be easily solved. The roots of the characteristic equation are equal to $\pm\mathrm{i}(1 \pm \kappa)$, where

$$\kappa = \tfrac{1}{2}\sqrt{\mu^2 + \nu^2} \tag{16}$$

and we neglect corrections of the order of κ^2. Calculating the elements of matrix λ_3 we neglect all terms of the order of μ, ν, κ in the amplitude coefficients (thus replacing matrix b_1 in the initial condition (15) by the unit matrix). After some algebra we find

$$\lambda_3 = \begin{Vmatrix} -\sin t\cos\tau & \sin\tau\sin(t-\varphi) \\ -\sin\tau\sin(t+\varphi) & -\sin t\cos\tau \end{Vmatrix}, \tag{17}$$

where $\tau \equiv \kappa t$ is a 'long time' and phase φ is defined via the relations

$$\mu = 2\kappa\sin\varphi, \qquad \nu = 2\kappa\cos\varphi. \tag{18}$$

With the same accuracy, $\lambda_4 = -\dot{\lambda}_3$, where the dot means differentiation with respect to t for fixed τ:

$$\lambda_4 = \begin{Vmatrix} \cos t\cos\tau & -\sin\tau\cos(t-\varphi) \\ \sin\tau\cos(t+\varphi) & \cos t\cos\tau \end{Vmatrix}. \tag{19}$$

There is no need to calculate matrix λ_1 independently, since for any quadratic Hamiltonian with time-independent coefficients the identities $\lambda_1(t) \equiv \tilde{\lambda}_4(-t)$ and $(\lambda_1\lambda_3^{-1})(t) \equiv -(\lambda_3^{-1}\lambda_4)(-t)$ hold [20]. In the case involved $\lambda_4(t) = \lambda_1(t)$ due to the specific form of matrix (19). The inverse matrix λ_3^{-1} reads

$$\lambda_3^{-1} = \Delta^{-1}\begin{Vmatrix} -\sin t\cos\tau & -\sin\tau\sin(t-\varphi) \\ \sin\tau\sin(t+\varphi) & -\sin t\cos\tau \end{Vmatrix},$$

where

$$\Delta \equiv \det\lambda_3 = \sin^2 t - \sin^2\varphi\sin^2\tau. \tag{20}$$

3.2. Propagators and state exchange

Putting explicit expressions for the λ-matrices in (8) we obtain the following form of the propagator of two resonantly coupled identical oscillators:

$$G(x_1, x_2, x_1', x_2'; \tau, t) = \frac{\mathrm{i}\mathcal{M}}{2\pi\sqrt{\Delta}}\exp\left\{\frac{\mathrm{i}}{4\Delta}\big[(x_1^2 + x_2'^2)\times(\sin 2t - \sin^2\tau\sin 2\varphi) + (x_2^2 + x_1'^2)(\sin 2t + \sin^2\tau\sin 2\varphi) - 2\sin 2\tau\sin\varphi(x_1x_2 + x_1'x_2') - 4\sin t\cos\tau(x_1x_1' + x_2x_2') + 4\sin\tau[x_2x_1'\sin(t+\varphi) - x_1x_2'\sin(t-\varphi)]\big]\right\}. \tag{21}$$

Here $\mathcal{M} = \pm 1$ is the so-called Maslov index, which depends on delicate details of the evolution: one has to determine the sign of the square root in the pre-exponential factor, especially

when the argument of this root is negative. However, the actual value of $\mathcal{M}$ is not important for our purposes. For $\kappa \ll 1$, t and τ can be considered as effectively independent variables, since an increment of t by $\pi/2$, for example, which completely changes the functions $\sin t$ and $\cos t$, does not lead to any appreciable change of the functions $\sin\tau$ and $\cos\tau$. So, one may consider τ in the expressions for λ-matrices and in the propagator as a slowly varying parameter.

For arbitrary values of t, τ and φ the quadratic form in the argument of propagator (21) contains all possible products of the arguments, so the factorization of the propagator in the form (7) is impossible. However, there are two special cases. The first one is $\sin\tau = 0$, but this is trivial, because function (21) factorizes into a product of two independent propagators of each oscillator, without any interaction between them. A nontrivial situation occurs if $\cos\tau = 0$, i.e. $\tau = \tau_n = \pi(n + 1/2)$, $n = 0, 1, \ldots$. Then all λ-matrices become anti-diagonal, resulting in the propagator (the 'short time' t may be arbitrary)

$$G(x_1, x_2, x_1', x_2'; \tau_n, t) = \frac{\mathrm{i}\mathcal{M}}{2\pi\sqrt{\sin(t-\varphi)\sin(t+\varphi)}} \times \exp\left\{\frac{\mathrm{i}}{2\sin(t+\varphi)}[\cos(t+\varphi)(x_1^2 + x_2'^2) - 2\upsilon_n x_1 x_2'] + \frac{\mathrm{i}}{2\sin(t-\varphi)}[\cos(t-\varphi)(x_2^2 + x_1'^2) + 2\upsilon_n x_1' x_2]\right\}. \quad (22)$$

Here $\upsilon_n = \sin\tau_n = (-1)^n$.

One can immediately recognize in expression (22) a product of two well known [46, 47] free propagators of a harmonic oscillator. Consequently, the two-mode propagator (22) has indeed the form (7) with $t_1 = t - \varphi$, $t_2 = t + \varphi$, $s_2 = -s_1 = \upsilon_n$, which means that a state exchange of the second kind takes place at $\tau = \tau_n$ for any initial states and coupling constants. As to the state exchange of the first kind, it can happen only provided the difference $t_2 - t_1 = 2\varphi$ is zero or a multiple of π (due to the periodicity of the harmonic oscillator motion).

(1) If $\varphi = 0$ or π (i.e. $\mu = 0$, $\cos\varphi = \pm 1$), then the propagator (22) can assume the form of a product of two delta functions at the moments of time $t_m = m\pi$ (since $t \approx \pi(n + 1/2)/\kappa \gg 1$, the integer m must be large enough). In this case

$$\psi(x_1, x_2; t_m) = \mathcal{N}\psi_1(\mp\upsilon_n\theta_m x_2)\psi_2(\pm\upsilon_n\theta_m x_1) \quad (23)$$

where $\theta_m = (-1)^m$. Note that the signs of arguments in the functions ψ_1 and ψ_2 always turn out to be opposite.

(2) In the case of $\varphi = \pi/2$ (i.e. $\nu = 0$) we have

$$\psi(x_1, x_2; \tilde{t}_m) = \tilde{\mathcal{N}}\psi_1(-\upsilon_n\theta_m x_2)\psi_2(-\upsilon_n\theta_m x_1), \quad (24)$$

where $t = \tilde{t}_m = \pi(m + 1/2)$ and θ_m was defined above. In this case the signs of arguments in the functions ψ_1 and ψ_2 are the same.

In order to see what happens in the generic case, when $\cos\tau$ can assume arbitrary values, we consider two special cases, when formula (21) can be simplified.

(a) If $\varphi = 0$ (i.e. $\mu = 0$), then

$$G(x_1, x_2, x_1', x_2'; \tau, t; \varphi = 0) = \frac{\mathrm{i}\mathcal{M}}{2\pi\sin t} \times \exp\left\{\frac{\mathrm{i}}{2\sin t}\big[\cos t(x_1^2 + x_2'^2 + x_2^2 + x_1'^2) - 2\cos\tau(x_1x_1' + x_2x_2') + 2\sin\tau(x_2x_1' - x_1x_2')\big]\right\}. \quad (25)$$

In particular, at times when $\sin t = 0$ and $\cos t = 1$ (for definiteness) the propagator (25) is factorized in the product of two delta-functions, but their arguments do not satisfy the condition of the state exchange of the first kind (5):

$$G(t = 2n\pi; \varphi = 0) = \mathcal{M}\delta(x_1' - \cos\tau x_1 + \sin\tau x_2) \times\delta(x_2' - \cos\tau x_2 - \sin\tau x_1).$$

Consequently,

$$\psi_{\varphi=0}^{t=2n\pi}(x_1, x_2; \tau) = \mathcal{M}\psi_1(\cos\tau x_1 - \sin\tau x_2) \times\psi_2(\cos\tau x_2 + \sin\tau x_1). \quad (26)$$

(b) If $\varphi = \pi/2$ (i.e. $\nu = 0$), then

$$G(\tau, t; \varphi = \pi/2) = \frac{\mathrm{i}\mathcal{M}}{2\pi\sqrt{\sin(t-\tau)\sin(t+\tau)}} \times \exp\left\{\frac{\mathrm{i}}{4\sin(t+\tau)\sin(t-\tau)} \times\big[\sin 2t(x_1^2 + x_2'^2 + x_2^2 + x_1'^2) - 2\sin 2\tau(x_1x_2 + x_1'x_2') + 4\sin\tau\cos t(x_1x_2' + x_1'x_2) - 4\sin t\cos\tau(x_1x_1' + x_2x_2')\big]\right\}.$$

If moreover $t - \tau = 2m\pi$, then one delta-function arises:

$$G(t - \tau = 2m\pi; \varphi = \pi/2) = \frac{\mathcal{M}\delta(x_1 - x_2 - x_1' + x_2')}{\sqrt{\mathrm{i}\pi\sin 2\tau}} \times \exp\left(\frac{\mathrm{i}}{4\sin 2\tau}\big\{\cos 2\tau[(x_1 + x_2)^2 + (x_1' + x_2')^2] - 2(x_1 + x_2)(x_1' + x_2')\big\}\right).$$

For $\tau = n\pi$ we find $\mathcal{N}\delta(x_1 - x_1')\delta(x_2 - x_2')$, whereas for $\tau = (n + 1/2)\pi$ we obtain again $\mathcal{N}\delta(x_1 + x_2')\delta(x_2 + x_1')$.

Introducing bosonic annihilation operators

$$a_1 = \frac{x_1 + \mathrm{i}p_1}{\sqrt{2}}, \qquad a_2 = \frac{x_2 + \mathrm{i}p_2}{\sqrt{2}}$$

one can write the interaction part of Hamiltonian (1) as

$$2H_{\mathrm{int}} = (\gamma_1 + \gamma_4 + \mathrm{i}\gamma_2 - \mathrm{i}\gamma_3)\, a_1^\dagger a_2 + (\gamma_1 + \gamma_4 - \mathrm{i}\gamma_2 + \mathrm{i}\gamma_3)\, a_2^\dagger a_1 + (\gamma_4 - \gamma_1 - \mathrm{i}\gamma_2 - \mathrm{i}\gamma_3)\, a_2 a_1 + (\gamma_4 - \gamma_1 + \mathrm{i}\gamma_2 + \mathrm{i}\gamma_3)\, a_2^\dagger a_1^\dagger.$$

The first two terms can be written as $(\mu + \mathrm{i}\nu)a_1^\dagger a_2 + \mathrm{h.c.}$, and they correspond to the rotating wave approximation of the interaction Hamiltonian. Since these terms usually describe scattering effects, it is not surprising that they are responsible for the state exchange. It is interesting that only pure real or pure imaginary coefficients at the RWA terms permit the quantum state exchange of the first kind for any initial state. Moreover, the state exchange properties are different for the cases when $\mu = 0$ or $\nu = 0$. As to the 'counter-rotating' (rapidly oscillating) terms $a_1 a_2$ and $a_1^\dagger a_2^\dagger$, their contribution turns out to be nonessential in the weak-coupling limit.

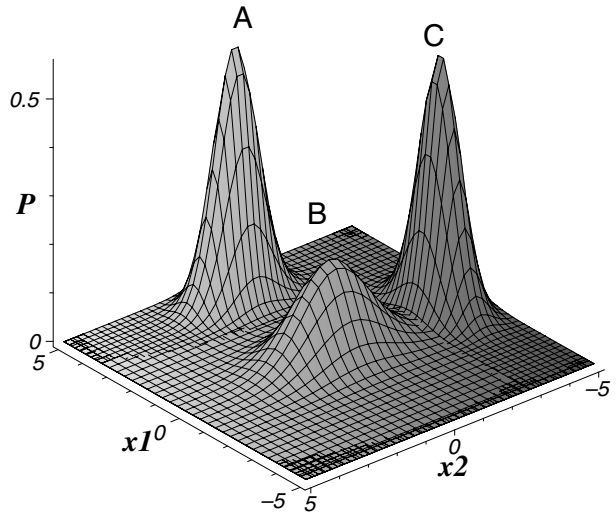

Figure 1. The exchanges of the first (C) and the second (B) kinds between the vacuum state and the squeezed state (27) with $s = 0.36$ and $x_0 = 3.86$. A: the initial probability density; C: the probability density at the instant $t = 157.08 \approx 50\pi$ ($\tau = \pi/2$) for $\varphi = \mu = 0$ and $\nu = 0.02$; B: the probability density for $\varphi = \pi/4$ ($\mu = \nu = 0.02$) at the instant $t = (35 - 1/4)\pi \approx 109.17$.

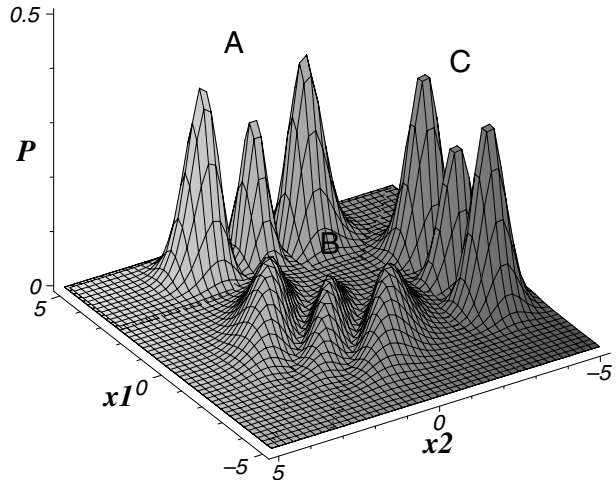

Figure 2. The evolution of the initial distribution for $\varphi = 0$ and $\nu = 0.02$. A: the first oscillator in the squeezed state (27) with $s = 0.36$, $x_0 = 3.85$, the second oscillator in the Fock state (29) with $n = 2$; B: the probability density for an arbitrary choice of time $t = 67.32 \approx 21.43\pi$ ($\tau \approx \pi/5$); C: the state exchange of the first kind for $t = 157.08 \approx 50\pi$ and $\tau \approx \pi/2$.

3.3. Illustrations

Figures 1–4 show the evolution of the joint probability density $P = |\Psi(x_1, x_2, t)|^2$. In all plots, the symbol A indicates the initial quantum state, whereas C corresponds to the instants of time when the state exchange of the first kind happens. In the examples considered, the nonzero values of μ or/and ν are chosen to be equal to 0.02, so that $\kappa = 0.010$ (i.e. $\tau = 0.010t$ in dimensionless units $m = \omega = \hbar = 1$) in the case of $\varphi = 0$, while $\kappa = 0.014$ in the case of $\varphi = \pi/4$. Making numerical calculations, we did not assume that τ and t were independent variables.

In all examples the first oscillator is initially in the displaced squeezed state

$$\psi^s_{x_0}(x) = (\pi\, s)^{-1/4} \exp[-(x - x_0)^2/(2s)] \qquad (27)$$

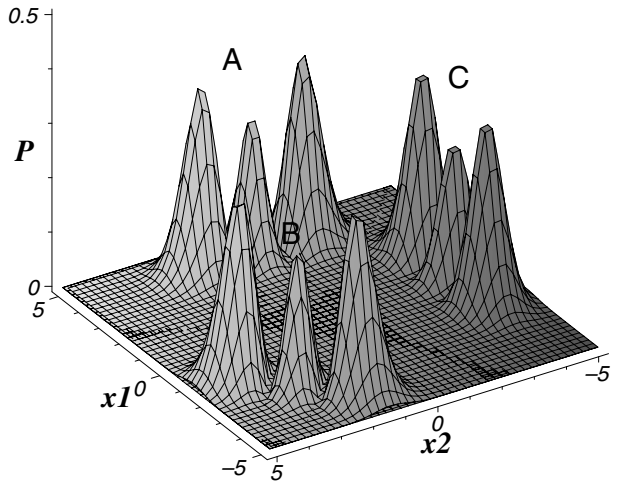

Figure 3. The evolution of the initial distribution for $\varphi = 0$ and $\nu = 0.02$. A: the first oscillator in the squeezed state (27) with $s = 0.36$, $x_0 = 3.85$, the second oscillator in the Fock state (29) with $n = 2$; B: a strongly entangled 'rotated' state of two modes at $t = 78.54 \approx 25\pi$ and $\tau \approx \pi/4$; C: the state exchange of the first kind for $t = 157.08 \approx 50\pi$ and $\tau \approx \pi/2$.

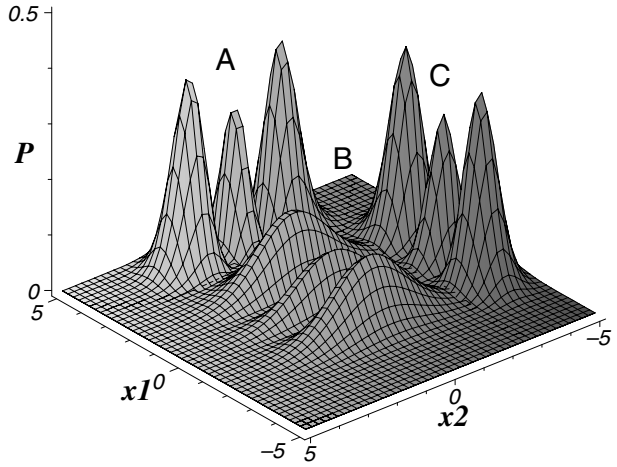

Figure 4. The exchanges of the first (C) and the second (B) kinds between the squeezed state (27) with $s = 0.36$, $x_0 = 3.85$ and the Fock state $|n\rangle$ (29) with $n = 2$, for $\varphi = \pi/4$ ($\mu = \nu = 0.02$). A: the initial probability density; B: the exchange of the second kind at $t = (36 - 1/4)\pi \approx 112.31$ ($\tau \approx 1.57$); C: the 'accidental' exchange of the first kind at $t = (36 + 1/4)\pi \approx 113.88$.

with $s = 0.36$ and $x_0 = 3.86$. Figure 1 describes the case when the second oscillator is initially in the vacuum state

$$\psi_0(x) = \pi^{-1/4} \exp(-x^2/2). \qquad (28)$$

For $\varphi = \mu = 0$ an exchange of quantum states of the first kind is observed at the instant $t = 157.08 \approx 50\pi$, when $\tau = \pi/2$. The probability density at this instant is given by the surface C, which is obviously the initial surface A rotated by 90°. Note that the coordinate of the centre of the squeezed packet changes its sign in full accordance with (23). Surface B shows the state exchange of the second kind for $\varphi = \pi/4$ (i.e. $\mu = \nu > 0$): at the instant $t = (35 - 1/4)\pi \approx 109.17$ ($\tau \approx 1.54$) the first oscillator occurs in the vacuum state, whereas the second oscillator goes to the squeezed state with the coordinate squeezing coefficient $s' = 1/s$ and the centre

of the packet shifted to the origin of the coordinate system. This change corresponds perfectly to the initial squeezed state shifted in time by a quarter of a period. If we made the plot in the momentum representation, it would coincide with the initial surface A rotated by 90° (like C, but located at the centre).

Figures 2–4 show different cases when the second oscillator is initially in the Fock state

$$\langle x|n\rangle = (2^n n!)^{-1/2}\psi_0(x)H_n(x) \tag{29}$$

($H_n(x)$ is the Hermite polynomial) with $n = 2$. Figures 2 and 3 show the case of $\varphi = \mu = 0$. An exchange of the first kind (with reflection of one coordinate) is observed at $t = 157.08 \approx 50\pi$, when $\tau = 1.57$ (surface C). Surface B in figure 2 corresponds to a more or less arbitrary choice of time $t = 67.32 \approx 21.43\pi$ ($\tau \approx \pi/5$). We see a joint distribution having a little in common with the initial one (excepting the number of peaks). In figure 3, surface B corresponds to the choice $t = 78.54 \approx 25\pi$ ($\tau \approx \pi/4$), when the probability density of strongly entangled modes is very close to the initial one, but rotated by 45° in the x_1–x_2 plane, in accordance with equation (26).

Figure 4 corresponds to $\varphi = \pi/4$ and $\tau \approx \pi/2$. For $t = (36 - 1/4)\pi \approx 112.31$ the state exchange of the second kind is observed (surface B): the wavefunction of the second oscillator becomes stretched in the x_2-direction with the 'squeezing' coefficient $s' = s^{-1} \approx 2.78$ (and the centre of the packet occurs at $x_2 = 0$, as it must for the time shift by a quarter of period). Unexpectedly, at first sight, we see a clear 'total' exchange (with the change of sign of one coordinate) for $t = (36+1/4)\pi \approx 113.88$ (surface C). This happens due to the specific property of the Fock state which, being an eigenstate of the free oscillator Hamiltonian, only changes its phase in the process of free evolution.

4. Time-dependent coupling

4.1. Solving equations for λ-matrices

If the frequencies ω_1 and ω_2 are different, the resonance can occur if the coupling constants γ_j are periodic functions of time, with the frequency $\eta = \omega_1 - \omega_2$. Therefore, let us consider the time-dependent Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2) + \cos(\eta t) \times(\gamma_1 p_1 p_2 + \gamma_2 p_1 x_2 + \gamma_3 x_1 p_2 + \gamma_4 x_1 x_2).$$

Introducing the notation $\Gamma(t) \equiv \cos(\eta t)$ we can write the matrices $b_j(t)$ as

$$b_1 = \begin{Vmatrix} 1 & \gamma_1\Gamma(t) \\ \gamma_1\Gamma(t) & 1 \end{Vmatrix}, \qquad b_4 = \begin{Vmatrix} \omega_1^2 & \gamma_4\Gamma(t) \\ \gamma_4\Gamma(t) & \omega_2^2 \end{Vmatrix},$$

$$b_2 = \tilde{b}_3 = \begin{Vmatrix} 0 & \gamma_2\Gamma(t) \\ \gamma_3\Gamma(t) & 0 \end{Vmatrix}.$$

The second-order equation for matrix λ_3 (and the same one for λ_1) is

$$\frac{\mathrm{d}^2\lambda_3}{\mathrm{d}t^2} - \frac{\mathrm{d}\lambda_3}{\mathrm{d}t}\mathcal{E}_3 + \lambda_3\mathcal{E}_4 = 0, \tag{30}$$

where

$$\mathcal{E}_3 = b_3 - b_1^{-1}b_2b_1 + b_1^{-1}\frac{\mathrm{d}b_1}{\mathrm{d}t},$$

$$\mathcal{E}_4 = b_4b_1 - b_3b_1^{-1}b_2b_1 + b_3b_1^{-1}\frac{\mathrm{d}b_1}{\mathrm{d}t} - \frac{\mathrm{d}b_3}{\mathrm{d}t}.$$

In turn, each row of matrix λ_3 (or λ_1) can be considered as a two-dimensional vector $\boldsymbol{v} = (\lambda_{l1}, \lambda_{l2}) \equiv (v_1, v_2)$ (where $l = 1, 2$). Therefore the matrix equation (30) is equivalent to the vector equation

$$\ddot{\boldsymbol{v}} + \mathcal{M}\boldsymbol{v} = \mathcal{N}(t)\dot{\boldsymbol{v}} + \mathcal{Q}(t)\boldsymbol{v} \tag{31}$$

with diagonal matrix $\mathcal{M} = \mathrm{diag}\left(\omega_1^2, \omega_2^2\right)$ on the left-hand side and antidiagonal matrices on the right-hand side:

$$\mathcal{N} = \begin{Vmatrix} 0 & h_1 \\ h_3 & 0 \end{Vmatrix}, \qquad \mathcal{Q} = \begin{Vmatrix} 0 & -h_2 \\ -h_4 & 0 \end{Vmatrix},$$

where (evidently, $\dot{\Gamma}(t) \equiv -\eta\sin(\eta t)$)

$$h_1 = (\gamma_2 - \gamma_3)\Gamma(t) + \gamma_1\dot{\Gamma}(t),$$

$$h_2 = (\gamma_4 + \omega_2^2\gamma_1)\Gamma(t) - \gamma_2\dot{\Gamma}(t),$$

$$h_3 = (\gamma_3 - \gamma_2)\Gamma(t) + \gamma_1\dot{\Gamma}(t),$$

$$h_4 = (\gamma_4 + \omega_1^2\gamma_1)\Gamma(t) - \gamma_3\dot{\Gamma}(t).$$

Here we have neglected the terms proportional to squares and products of the coupling constants.

Since the coefficients γ_j are supposed to be small, we can use the Bogoliubov–Mitropolsky scheme [48] of finding approximate analytical solutions to equation (31). Thus we introduce amplitudes z_j and z_{-j} slowly varying in time according to the relations

$$v_j = z_j\mathrm{e}^{\mathrm{i}\omega_j t} + z_{-j}\mathrm{e}^{-\mathrm{i}\omega_j t}, \tag{32}$$

$$\dot{v}_j = \mathrm{i}\omega_j z_j\mathrm{e}^{\mathrm{i}\omega_j t} - \mathrm{i}\omega_j z_j\mathrm{e}^{-\mathrm{i}\omega_j t}. \tag{33}$$

Putting these expressions into equation (31) and using the method of averaging (i.e. multiplying the equations by the factors $\exp(\pm\mathrm{i}\omega_j t)$ and averaging the arising expressions over the period of fast oscillations with frequency ω_j), one can obtain for functions $z_j(t)$ and $z_{-j}(t)$ a set of the first-order linear differential equations with constant coefficients, all of which have an order of (small) coupling parameters γ_k. This method works just because the interaction Hamiltonian is a periodic function of time with the frequency η, which is equal exactly to the difference $\omega_1 - \omega_2$. In principle, one could consider a more general resonance interaction Hamiltonian, with coupling coefficients depending on time as $\gamma_j\cos(\eta t - \phi_j)$ and with different phases ϕ_j (but with the same frequency η). However, here we confine ourselves to the simplest case of equal phases. The choice of the periodic dependence in the form of $\cos(\eta t)$ is justified by the aim of reducing the time-dependent interaction Hamiltonian to that considered in the preceding section for $\eta = 0$.

Omitting all details of cumbersome calculations, we give only the final answer: the explicit forms of elements of matrices λ_j which are necessary to construct the propagator (8):

$$\lambda_1^{11} = \cos\tilde{\tau}\cos\omega_1 t, \qquad \lambda_1^{12} = -\sqrt{\frac{\omega_1}{\omega_2}}\sin\tilde{\tau}\cos(\omega_2 t - \varphi),$$

$$\lambda_1^{21} = \sqrt{\frac{\omega_2}{\omega_1}}\sin\tilde\tau\cos(\omega_1 t+\varphi), \qquad \lambda_1^{22} = \cos\tilde\tau\cos\omega_2 t,$$

$$\lambda_3^{11} = -\cos\tilde\tau\frac{\sin\omega_1 t}{\omega_1}, \qquad \lambda_3^{12} = \frac{\sin\tilde\tau}{\sqrt{\omega_1\omega_2}}\sin(\omega_2 t-\varphi),$$

$$\lambda_3^{21} = -\frac{\sin\tilde\tau}{\sqrt{\omega_1\omega_2}}\sin(\omega_1 t+\varphi), \qquad \lambda_3^{22} = -\cos\tilde\tau\frac{\sin\omega_2 t}{\omega_2},$$

$$\lambda_4^{11} = \cos\tilde\tau\cos\omega_1 t, \qquad \lambda_4^{12} = -\sqrt{\frac{\omega_2}{\omega_1}}\sin\tilde\tau\cos(\omega_2 t-\varphi),$$

$$\lambda_4^{21} = \sqrt{\frac{\omega_1}{\omega_2}}\sin\tilde\tau\cos(\omega_1 t+\varphi), \qquad \lambda_4^{22} = \cos\tilde\tau\cos\omega_2 t.$$

Here

$$\sin\varphi = \frac{\mu_0}{\sqrt{\mu_0^2+\nu_0^2}}, \qquad \cos\varphi = \frac{\nu_0}{\sqrt{\mu_0^2+\nu_0^2}}, \tag{34}$$

$$\mu_0 = \omega_1\omega_2\gamma_1+\gamma_4, \qquad \nu_0 = \omega_1\gamma_2-\omega_2\gamma_3. \tag{35}$$

The 'long-time' dimensionless parameter $\tilde\tau$ is now given by

$$\tilde\tau = \frac{t\sqrt{\mu_0^2+\nu_0^2}}{4\sqrt{\omega_1\omega_2}} \equiv \tilde\kappa t. \tag{36}$$

The matrices whose elements are given above satisfy the initial conditions and the symplectic conditions for matrix Λ [19,20]

$$\lambda_4(t)\tilde\lambda_1(t) - \lambda_3(t)\tilde\lambda_2(t) = E_2,$$

$$\lambda_3(t)\tilde\lambda_4(t) = \lambda_4(t)\tilde\lambda_3(t), \qquad \lambda_1(t)\tilde\lambda_3(t) = \lambda_3(t)\tilde\lambda_1(t).$$

Note that for $\omega_1 = \omega_2 = 1$ parameter $\tilde\tau$ turns out to be half the 'long time' $\tau = \kappa t$ used in the preceding section for time-independent coupling constants. This difference can be explained as follows. Each time-dependent coefficient $\gamma\cos(\eta t)$ can be represented as a sum of two complex exponentials $\frac{1}{2}\gamma[\exp(\mathrm{i}\eta t) + \exp(-\mathrm{i}\eta t)]$. Therefore, calculating average values of the derivatives $\dot z_j$ one meets, among other terms, the combinations such as

$$\tfrac{1}{2}\gamma[\exp(\mathrm{i}[\eta-\omega_1+\omega_2]t) + \exp(-\mathrm{i}[\eta+\omega_1-\omega_2]t)].$$

As long as we assume $\eta = \omega_1 - \omega_2$, the first term in the expression above survives after averaging, while the second term, proportional to $\exp(-2\mathrm{i}\eta t)$, is supposed to go to zero in the process of averaging. Thus the contribution from the terms involved to the averaged equation of motion is equal to $\frac{1}{2}\gamma$, whereas in the case of equal frequencies it was equal to γ (because the terms $\exp(\mathrm{i}\eta t)$ and $\exp(-\mathrm{i}\eta t)$ are identical for $\eta = 0$), but the term $\exp(-2\mathrm{i}\eta t)$ disappears after averaging over many periods of 'fast' oscillations only under the condition that η has the same order of magnitude as frequencies ω_1 and ω_2, whereas for $\eta \ll \omega_{1,2}$ this term should be taken into account. Since averaging time cannot exceed $\tilde\kappa^{-1}$ (because the long time $\tilde\tau = \tilde\kappa t$ should not change appreciably during averaging), the solutions given above are justified provided $\eta \gg \tilde\kappa$. Consequently, a direct transition from the case of $\omega_1 - \omega_2 \neq 0$ to the case of $\omega_1 - \omega_2 = 0$ is impossible in the framework of the simplest version of the Bogoliubov–Mitropolskiy scheme used here, but it requires a special study, which is not considered here. It is remarkable, nonetheless, that for $\omega_1 = \omega_2 = 1$ the expressions for elements of λ-matrices given above formally coincide with the solutions (17) and (19) if one replaces $\tilde\tau$ by τ.

4.2. Propagators and state exchange

In order to simplify the formulae, it is convenient to introduce the scaled coordinates $y_j = \sqrt{\omega_j}\,x_j$ (in the dimensionless units $y_j = \sqrt{m_j\omega_j/\hbar}\,x_j$). Then the weak-coupling resonance propagator can be expressed as follows:

$$\begin{aligned} G(y_1, y_2, y_1', y_2'; \tilde\tau, t) = {} & \frac{\mathrm{i}\mathcal{M}}{2\pi\sqrt{\Delta_\eta}}\exp\left\{\frac{\mathrm{i}}{2\Delta_\eta}\right. \\ & \times\big[S_2(t)C_1(t)(y_1^2+y_1'^2) + S_1(t)C_2(t)(y_2^2+y_2'^2) \\ & + \sin^2\tilde\tau[\sin\varphi\cos\varphi_\eta(y_2^2-y_1^2) + \cos\varphi\sin\varphi_\eta(y_1'^2-y_2'^2)] \\ & + 2\sin\tilde\tau[y_2y_1'\sin(\omega_1 t+\varphi) - y_1y_2'\sin(\omega_2 t-\varphi)] \\ & - \sin 2\tilde\tau(\sin\varphi y_1y_2 + \sin\varphi_\eta y_1'y_2') \\ & \left. -2\cos\tilde\tau[S_2(t)y_1y_1' + S_1(t)y_2y_2']\big]\right\}, \end{aligned} \tag{37}$$

where

$$\Delta_\eta = \omega_1\omega_2\det\lambda_3 = S_1S_2 - \sin^2\tilde\tau\sin\varphi\sin\varphi_\eta, \tag{38}$$

$$S_j = \sin\omega_j t, \qquad C_j = \cos\omega_j t, \qquad j = 1, 2,$$
$$\varphi_\eta = \varphi + \eta t.$$

As in the case of identical frequencies, the state exchange in a wide sense can occur for arbitrary initial states if $\cos\tilde\tau = 0$, i.e. for $\tilde\tau = \tilde\tau_n = \pi(n+1/2)$, $n = 0, 1, \ldots$. Under this condition the propagator can be written as (with the same meaning of $\mathcal{M}$ and υ_n as before)

$$\begin{aligned} G(y_1, y_2, y_1', y_2'; \tilde\tau_n, t) = {} & \frac{\mathcal{M}}{2\pi\mathrm{i}\sqrt{\sin(\omega_2 t-\varphi)\sin(\omega_1 t+\varphi)}} \\ & \times\exp\left\{-\frac{\cos(\omega_1 t+\varphi)(y_1^2+y_2'^2) - 2\upsilon_n y_1y_2'}{2\mathrm{i}\sin(\omega_1 t+\varphi)}\right. \\ & \left. -\frac{\cos(\omega_2 t-\varphi)(y_2^2+y_1'^2) + 2\upsilon_n y_1'y_2}{2\mathrm{i}\sin(\omega_2 t-\varphi)}\right\}, \end{aligned} \tag{39}$$

so it has the factorized form (7) with the scaling factors $s_2 = -s_1 = \upsilon_n$ and effective times

$$t_1 = \frac{\omega_2}{\omega_1}t - \frac{\varphi}{\omega_1}, \qquad t_2 = \frac{\omega_1}{\omega_2}t + \frac{\varphi}{\omega_2}.$$

As to the state exchange in a narrow sense, this can happen provided the ratio of frequencies is adjusted with coupling coefficients. Indeed, the propagator (39) turns into a product of two delta-functions at the instant t_* if

$$\omega_1 t_* = (q-\chi)\pi, \qquad \omega_2 t_* = (p+\chi)\pi,$$

where p, q are integers and $\chi \equiv \varphi/\pi$, $-1 \leqslant \chi \leqslant 1$. Consequently,

$$t_* = \frac{p+q}{\omega_1+\omega_2}\pi, \qquad \frac{\omega_1}{\omega_2} = \frac{q-\chi}{p+\chi}. \tag{40}$$

In particular, for $\chi = 0$ (when $\mu_0 = 0$) the ratio ω_1/ω_2 can be any rational number. However, for $\chi = \pm 1/2$ (when $\nu_0 = 0$) we obtain $\omega_1/\omega_2 = (2q-1)/(2p+1)$. Therefore, although the ratio still must be a rational number, it cannot be arbitrary. For example, even values $\omega_1/\omega_2 = 2, 4, \ldots$ must be excluded. On the other hand, one can obtain the state exchange of the first kind in the case when both coefficients, μ and ν, are different from zero, choosing proper values of frequencies.

For example, for $\mu = \nu > 0$ (when $\chi = 1/4$) the frequencies must satisfy the relation $\omega_1/\omega_2 = (4q-1)/(4p+1)$. Actually, natural numbers p and q must be sufficiently large, of the order of $[\omega_1\omega_2/(\mu_0^2+\nu_0^2)]^{1/2}$. However, it is easy to show that if some pair of integers p_0, q_0 is found, then an infinite number of other pairs exists, provided χ is a rational number: $\chi = a/b$ with integral a and b. A general formula describing such pairs is

$$p = n(bp_0 + a) + p_0, \qquad q = n(bq_0 - a) + q_0,$$
$$n = 0, 1, 2, \ldots.$$

For irrational values of χ (and ω_1/ω_2), only one pair of integers p, q can be found, unless χ is approximated by some rational number. One should bear in mind, however, that even for rational χ the state exchange of the first kind can be, strictly speaking, only approximate, with rare accidental exceptions. This is because 'long time' τ (or $\tilde{\tau}$) in reality is not completely independent from the real time t, therefore conditions $\tau = (n + 1/2)\pi$ and $t = t_*$ can be satisfied exactly and simultaneously only under very special circumstances. In most cases the state exchange of the first kind can occur in the form of example (26) from section 3, with small value of $\cos\tau$. Diminishing coupling constants, one can obtain better conditions for the state exchange. However, the time necessary for the exchange grows inversely proportional to coupling constants.

5. Conclusion

We have introduced two concepts of the state exchange between coupled modes of the electromagnetic field or coupled harmonic oscillators. The analysis of possibility of the exchange was reduced to the analysis of conditions under which the total quantum mechanical propagator is factorized in some specific form, which becomes a product of two specific delta-functions in the case of the state exchange in a narrow sense. (Note that factorization of propagators was studied from another point of view in [49]; factorization of the Wigner functions was considered in [50] and factorization of wavefunctions was considered in [51].) We have shown that the state exchange in the wide sense is always possible (in the resonance weak-coupling limit) in certain relatively short intervals of time, when the 'long-time' parameters τ or $\tilde{\tau}$ become sufficiently close to $(n + 1/2)\pi$. As to the state exchange in the narrow sense, this is possible under additional restrictions on the values of coupling coefficients and the ratio of mode frequencies. Couplings in the form of the 'rotating wave approximation' play a distinguished role for the possibility of the state exchange in the narrow sense. Moreover, 'coordinate–momentum' couplings turn out to be preferable to 'coordinate–coordinate' and 'momentum–momentum' couplings.

We should remark that nice explicit expressions for the propagators showing a possibility of the state exchange of the second and first kinds have been found in the framework of certain approximations: we have neglected corrections proportional to small coupling constants in the amplitude coefficients of the λ-matrices and neglected the contibution of terms with higher harmonics in the case of time-dependent resonance coupling. The influence of these small corrections will be considered elsewhere.

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