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Creating quanta with an 'annihilation' operator

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Abstract

The asymmetric nature of the boson 'destruction' operator $\hat{a}$ and its 'creation' partner $\hat{a}^{\dagger}$ is made apparent by applying them to a quantum state $|\psi\rangle$ different from the Fock state $|n\rangle$. We show that it is possible to *increase* (by many times or by any quantity) the mean number of quanta in the new 'photon-subtracted' state $\hat{a}|\psi\rangle$. Moreover, for certain 'hyper-Poissonian' states $|\psi\rangle$ the mean number of quanta in the (normalized) state $\hat{a}|\psi\rangle$ can be much greater than in the 'photon-added' state $\hat{a}^{\dagger}|\psi\rangle$. The explanation of this 'paradox' is given and some examples elucidating the meaning of Mandel's q-parameter and the exponential phase operators are considered.

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1. Introduction

The non-Hermitian bosonic operators $\hat{a}$ and $\hat{a}^{\dagger}$ of the harmonic oscillator, satisfying the canonical commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$, are usually called 'annihilation' [1] (or 'destruction' [2, 3][2]) and 'creation' operators (perhaps, only Dirac, in his book [4], did not use these terms, introducing instead of $\hat{a}$ and $\hat{a}^{\dagger}$ the 'complex dynamical variables' $\bar{\eta}$ and η). This is due to their action on the Fock (number) state $|n\rangle$ (eigenstate of the number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$) [5]

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{1}$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2}$$

Therefore, there is some belief that the operator $\hat{a}$ may 'destroy' quanta (photons) in an *arbitrary* state $|\psi\rangle$. This belief is reflected even in the name 'photon-subtracted state',

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[2] Page 27 in [3] attributes gods names, from Hindu mythology, to the operators: $\hat{a}$ is *Siva*, the Destroyer, $\hat{a}^{\dagger}$ is *Brahma*, the Creator and $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is *Vishnu*, the Preserver.

sometimes used for the state $\hat{a}|\psi\rangle$ [6–8]. However, this concept, guided maybe more by intuition than by a sound proof, seems to be misleading when one deals not with single Fock states, but with their superpositions or quantum mixtures.

The best-known counter-example is the harmonic oscillator coherent state $|\alpha\rangle$: since it is an eigenstate of $\hat{a}$, the state $\hat{a}|\alpha\rangle/|\alpha|$ has the same mean number of photons as $|\alpha\rangle$ (see a similar remark in [7]). But as we will see below, this example is not the only one. In contradistinction, when operator $\hat{a}^\dagger$ is applied on an arbitrary state $|\psi\rangle$ it produces a new one, whose mean number of quanta is always greater than that of $|\psi\rangle$, therefore justifying its designation as a 'creation operator'. Our aim is to provide a deeper investigation of the problem of 'photon subtraction' or 'photon addition' for *arbitrary* quantum states, drawing the reader's attention to apparent 'paradoxes' and giving possible explanations.

The plan of the paper is as follows. In section 2, we discuss the relations between the 'photon excess' in the 'photon-subtracted' states and Mandel's parameter, illustrating them in the example of superposition of two Fock states. The concept of 'hyper-Poissonian' states is introduced in section 3. The examples of such states include certain superpositions of the coherent and vacuum states and the negative binomial states. New families of 'logarithmic' states corresponding to the given values of the mean number of quanta and Mandel's parameter are considered in section 4. A distinguished role of the exponential phase operators and their eigenstates—the phase coherent states—is discussed in section 5. Section 6 concludes the paper.

2. 'Photon-subtracted' states, 'photon excess' and Mandel's parameter

For any normalized state

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \qquad \sum_{n=0}^{\infty} |c_n|^2 = 1 \tag{3}$$

the normalized 'photon-subtracted state' is given by

$$|\psi_-\rangle = \frac{1}{\sqrt{\bar{n}}}\hat{a}|\psi\rangle = \frac{1}{\sqrt{\bar{n}}}\sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \frac{1}{\sqrt{\bar{n}}}\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle \tag{4}$$

where

$$\bar{n} = \langle\psi|\hat{n}|\psi\rangle = \sum_{n=1}^{\infty} n|c_n|^2 \tag{5}$$

is the mean value of the quantum number operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ in $|\psi\rangle$. (In this paper, we consider mainly pure states, with the only aim to simplify notation.) Designating the mean number of quanta in the state $|\psi_-\rangle$ as

$$N_- = \langle\psi_-|\hat{n}|\psi_-\rangle = \frac{1}{\bar{n}}\sum_{n=1}^{\infty} n(n-1)|c_n|^2 \tag{6}$$

it is easy to see that the difference between N_- and $\bar{n}$ (the 'photon excess') is nothing other than Mandel's q-parameter [9]:

$$N_- - \bar{n} = \overline{n^2}/\bar{n} - 1 - \bar{n} \equiv q. \tag{7}$$

Note that the transformation $|\psi\rangle \to |\psi_-\rangle$ is not a pure mathematical exercise. It is a real consequence of conditional measurements on beam splitters [6, 10]. Moreover, according to the Srinivas–Davies theory of photodetection [11], the quantum states of field modes undergo the same transformation upon detection of one photon. In the last case, equation (7) and the related interpretation of the physical meaning of Mandel's parameter were discussed in [12, 13].

As follows from (7), the 'annihilation' operator $\hat{a}$ diminishes effectively the mean number of quanta ($q < 0$) only for states having *sub-Poissonian statistics* (which have always been considered as 'nonclassical'). There are many families of 'sub-Poissonian' states, besides the Fock ones. In particular, all 'binomial' states [14, 15] possess sub-Poissonian photon statistics. Other examples include the families of so-called 'Barut–Girardello states' [16, 17] or odd coherent states [18].

For any super-Poissonian state, the action of the 'annihilation' operator results in an *increase* in the mean number of quanta. The explanation of such 'paradoxical' behaviour stems from the first factor, $\sqrt{n}$, on the right-hand side of equation (1): besides shifting the Fock state $|n\rangle$ by one quantum to the left, the operator $\hat{a}$ also increases the amplitude of the new state. Therefore, under certain conditions the relative weight of Fock states with larger numbers of quanta in state (4) can be greater than in the initial state (3), which results finally in an increase in the mean number of quanta (cf [12]).

2.1. Superposition of two Fock states

Suppose the superposition of two different Fock states

$$|\psi\rangle_F = \sqrt{r}|n\rangle + \sqrt{1-r}|m\rangle \qquad 0 < r < 1. \tag{8}$$

Then

$$\bar{n} = rn + (1-r)m \qquad N_- = \frac{rn(n-1) + (1-r)m(m-1)}{rn + (1-r)m}$$

and the condition $N_- > \bar{n}$ is equivalent to the inequality $r(1-r)(n-m)^2 > r(n-m) + m$, which can be satisfied for any r (different from 0 and 1) if the difference $n - m$ is sufficiently large.

Since there are no limitations on the positive values of Mandel's parameter, applying the 'annihilation' operator to highly super-Poissonian states one can increase the mean number of quanta by any desired quantity. For example, in the case of state (8) one has $q \approx (1-r)(n-m)$ if $r(n-m) \gg m$, and one can obtain $q \gg 1$ if also $(1-r)(n-m) \gg 1$. On the other hand, by calculating the q-factor for the state $|\psi_-\rangle_F$, one can see that it tends to the limit value

$$q_-^\infty = \frac{1-r}{r}m - 1$$

when $n \to \infty$, for fixed m and r. So, if $(1-r)m \ll 1$, but $(1-r)n \gg 1$, then by applying the operator $\hat{a}$ on $|\psi\rangle$ one can significantly *increase* the mean number of quanta and simultaneously transform the highly super-Poissonian state $|\psi\rangle$ into sub-Poissonian state $|\psi_-\rangle$, which turns out to be much closer to the state $|n-1\rangle$ than the initial state was, with respect to $|n\rangle$. This happens due to a significant reduction of the relative weight of the low-energy state. Indeed, if initially this weight was equal to $(1-r)/r$, in the new state it becomes approximately $n/m \gg 1$ times smaller. In other words, the operator $\hat{a}$ effectively 'annihilates' the *low-energy components* of quantum superpositions, *increasing the relative weight* of the higher-energy components.

3. 'Hyper-Poissonian' states

Acting on the state $|\psi\rangle$ by the operator $\hat{a}^\dagger$ we obtain the 'photon-added state' [6–8, 19]

$$|\psi_+\rangle = \frac{1}{\sqrt{1+\bar{n}}}\hat{a}^\dagger|\psi\rangle = \frac{1}{\sqrt{1+\bar{n}}}\sum_{n=0}^{\infty} c_n\sqrt{1+n}|n+1\rangle. \tag{9}$$

The mean value of the quantum number operator in the state $|\psi_+\rangle$ equals

$$N_+ = \langle\psi_+|\hat{n}|\psi_+\rangle = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}(n+1)^2|c_n|^2 = \bar{n}+1+\frac{\overline{(\Delta n)^2}}{1+\bar{n}}. \tag{10}$$

The difference $N_+ - \bar{n}$ is always not less than 1 (being equal to 1 only for the Fock states). Therefore, the name 'creation operator' is justified for the operator $\hat{a}^\dagger$. Nevertheless, in certain cases one can add many more quanta by acting on state $|\psi\rangle$ not with the 'creation' operator, but with the 'annihilation' partner. Using the relation $\overline{(\Delta n)^2} = \bar{n}\,(1+q)$ together with equations (7) and (10), one can see that the condition $N_- > N_+$ is equivalent to

$$q > 1 + 2\bar{n}. \tag{11}$$

The states possessing the property $q > \bar{n}$ were named in [20, 21] as 'super-chaotic' or 'super-random' states (for such states, the second-order correlation parameter $g^{(2)}(0)$ is greater than its value for the thermal states $g^{(2)}_{\text{therm}}(0) = 2$). Therefore, the states possessing the property (11) can be named 'hyper-chaotic' or 'hyper-Poissonian'. Note that Mandel's parameter of the *squeezed vacuum states* is given exactly by the right-hand side of equation (11) (it can be calculated, e.g., using the results of [22]). Consequently, the 'hyper-Poissonian' states can be characterized as those for which the fluctuations of the photon number are stronger than in the squeezed vacuum states.

3.1. Superposition of coherent and vacuum states

Let us consider a superposition of the *coherent state* $|\alpha\rangle$ and the *vacuum state* $|0\rangle$

$$|\psi\rangle_{0\alpha} = \sqrt{\eta}|\alpha\rangle + \xi|0\rangle \qquad \eta + |\xi|^2 + 2\sqrt{\eta}\,\mathrm{e}^{-\alpha^2/2}\,\mathrm{Re}\,\xi = 1. \tag{12}$$

Since the overall phases are not essential, we assume that η and α are real positive numbers (whereas the coefficient ξ may be complex; however, it does not enter the formulae which we are interested in). In this case, the operator $\hat{a}$ 'annihilates' the vacuum component of the initial superposition (12), transforming $|\psi\rangle_{0\alpha}$ into the coherent state $|\psi_-\rangle_{0\alpha} = |\alpha\rangle$. This transformation is accompanied by an increase of the mean number of quanta under certain conditions:

$$\bar{n} = \eta\alpha^2 \qquad N_- = \alpha^2 \qquad N_+ = \frac{\eta\alpha^4 + 3\eta\alpha^2 + 1}{1+\eta\alpha^2}. \tag{13}$$

The ratio N_+/N_- equals

$$\frac{N_+}{N_-} = 1 - \frac{1-\alpha^{-2}-3\eta}{1+\eta\alpha^2}$$

thus for $3\eta + \alpha^{-2} < 1$ we have $N_+ < N_-$. Moreover, if $\alpha \gg 1$, but $\eta\alpha^2 \ll 1$, then $\bar{n} \ll 1 < N_+ \ll N_-$. The probabilities of finding n quanta in the superposition (12) are

$$p_0 = 1 - \eta\left(1-\mathrm{e}^{-\alpha^2}\right) \qquad p_n = \eta\,\mathrm{e}^{-\alpha^2}\frac{\alpha^{2n}}{n!}. \tag{14}$$

A special case of the state (12) without vacuum component ($p_0 = 0$) was considered in [23] as an example of 'the most nonclassical' state (see also a comment in [24]).

3.2. Generating functions and negative binomial states

It is well known that the statistical properties of the probability distribution $\{p_n \equiv |c_n|^2\}$ can be calculated with the aid of the generating function

$$G(z) = \sum_{n=0}^{\infty} p_n z^n \qquad p_n = \frac{1}{n!} G^{(n)}(0) \qquad G(1) = 1 \tag{15}$$

since its derivatives at $z = 1$ generate the factorial moments,

$$\mathrm{d}^r G(z)/\mathrm{d}z^r|_{z=1} = \overline{n(n-1)(n-2)\cdots(n-r+1)} \equiv n^{(r)}. \tag{16}$$

If the function $G(z)$ for the state $|\psi\rangle$ is known, then the functions $G_\pm(z)$ for the states $|\psi_\pm\rangle$ are given by the formulae

$$G_-(z) = \frac{1}{\bar{n}} \frac{\mathrm{d}G}{\mathrm{d}z} \qquad G_+(z) = \frac{1}{1+\bar{n}} z \frac{\mathrm{d}}{\mathrm{d}z}[zG(z)]. \tag{17}$$

Applying these relations to the super-Poissonian *negative binomial state* [25, 26]

$$|\xi, \mu\rangle = \sum_{n=0}^{\infty} \left[(1-\xi)^\mu \frac{\Gamma(\mu+n)}{\Gamma(\mu) n!} \xi^n \right]^{1/2} |n\rangle \qquad \mu > 0 \quad 0 \leqslant \xi < 1 \tag{18}$$

for which

$$q = \xi/(1-\xi) \qquad \bar{n} = \mu q \qquad G(z) = \left(\frac{1-\xi}{1-z\xi} \right)^\mu$$

we find

$$G_-(z) = \left(\frac{1-\xi}{1-z\xi} \right)^{\mu+1} \qquad G_+(z) = z \left(\frac{1-\xi}{1-z\xi} \right)^{\mu+1} \frac{1+z\xi(\mu-1)}{1+\xi(\mu-1)}.$$

Consequently, applying the operator $\hat{a}$ to the state (18) several times, one can add each time equal portions of quanta q ($N_- = \mu q + q = \bar{n} + q$) without changing Mandel's parameter ($q_- = q$). On the other hand, after applying the operator $\hat{a}^\dagger$ on $|\psi\rangle$ the mean number of quanta is increased,

$$N_+ - \bar{n} = 1 + \frac{\xi}{1-\xi} + \frac{\xi(\mu-1)}{1+\xi(\mu-1)}$$

and one can easily verify that for $\mu < 1/2 < \xi(1-\mu)$ the action of the 'annihilation' operator adds more quanta than the action of the 'creation' operator.

4. Quantum states with given mean photon number and Mandel's parameter

Suppose that we want to find a probability distribution $\{p_n\}$ with two given parameters: the mean number $\bar{n}$ and the 'mean photon multiplication factor' $\gamma \equiv N_-/\bar{n}$. Since $N_- = \mathrm{d}G_-/\mathrm{d}z|_{z=1}$, we can rewrite the second condition, using the first of the relations in equation (17), as $G''(1) = \gamma\bar{n}G'(1)$. Assuming that this relation must hold not only for $z = 1$, but also in some interval of values of the auxiliary variable z, including the point $z = 1$, we obtain a simple equation, $G''(z) = \gamma\bar{n}G'(z)$, whose solution, satisfying the conditions $G(1) = 1$ and $G'(1) = \bar{n}$, reads

$$G(z) = 1 - \frac{1}{\gamma} + \frac{1}{\gamma} \mathrm{e}^{\gamma\bar{n}(z-1)}. \tag{19}$$

The corresponding 'photon number' distribution is

$$p_0 = 1 - \frac{1}{\gamma} + \frac{1}{\gamma} \mathrm{e}^{-\gamma\bar{n}} \qquad p_n = \frac{1}{\gamma} \mathrm{e}^{-\gamma\bar{n}} \frac{(\gamma\bar{n})^n}{n!} \qquad n \geqslant 1. \tag{20}$$

A *pure* quantum state possessing the photon number probability distribution $\{p_n\}$ can be written as

$$|\psi\rangle = \sum_{n=0}^{\infty} e^{i\phi_n} \sqrt{p_n} |n\rangle \tag{21}$$

where the set of phases $\{\phi_n\}$ may be quite arbitrary. In particular, one can adjust the phases in such a way that function (21) together with the weight set (20) becomes a special case of the superposition (12), with $|\alpha|^2 = \gamma\bar{n}$ and $\eta = \gamma^{-1}$. The probability distribution (20) exists for any value of $\bar{n}$ if $\gamma \geqslant 1$. For $\gamma < 1$ (sub-Poissonian case) it has sense only for not very large values of the initial mean photon number: $\gamma\bar{n} \leqslant |\ln(1-\gamma)|$. For $\gamma \to 1$ the Poisson distribution is recovered, while for $\gamma \to \infty$, $p_0 = 1$ and all other probabilities go to zero.

Of course, there are infinitely many different distributions resulting in the same values of two parameters, $\bar{n}$ and γ or $\bar{n}$ and $q = (\gamma - 1)\bar{n}$. For example, one can start from the equation $G''(z) = \gamma\bar{n}^2 G(z)$, which results in the function

$$G(z) = \cosh[\bar{n}\sqrt{\gamma}(z-1)] + \gamma^{-1/2}\sinh[\bar{n}\sqrt{\gamma}(z-1)]. \tag{22}$$

However, this function can be interpreted as a generating function for nonnegative probabilities $\{p_n\}$ only under certain restrictions on the values of $\bar{n}$ and γ, because the coefficient p_1 in the Taylor expansion (15) of function (22) becomes negative for $\bar{n}\sqrt{\gamma} \gg 1$ and $\gamma > 1$, whereas for p_0 the same happens if $\bar{n}\sqrt{\gamma} \gg 1$ and $\gamma < 1$. This example shows the main difficulty in finding new distributions through differential equations for their generating functions: although one can write infinitely many differential equations for $G(z)$ which result in the necessary conditions at $z = 1$, only a few of them can be solved analytically, and even fewer functions thus obtained have *all* derivatives at $z = 0$ nonnegative. Nonetheless, several interesting distributions can be found in this way.

4.1. Generalized logarithmic states

For example, looking for probabilities resulting in a given value of the 'photon excess' (or Mandel's parameter) q, we can rewrite equation (7) (or $\overline{n^2} - \bar{n}^2 - (q+1)\bar{n} = 0$) as (using relations (16))

$$G''(1) - [G'(1)]^2 - qG'(1) = 0. \tag{23}$$

One of many possible extensions of this equality to a finite interval of values of variable z is the equation

$$G''(z) - [G'(z)]^2 - qG'(z) = 0. \tag{24}$$

Although nonlinear this equation is easily solved by several methods, for example, multiplying it on the left by the integrating factor e^{-G} one gets a linear equation $dF/dz = qF$ for $F = e^{-G}G'$. Taking into account the conditions $G(1) = 1$ and $G'(1) = \bar{n}$, we obtain

$$G(z) = 1 - \ln\left[1 + \frac{\bar{n}}{q}(1 - e^{q(z-1)})\right]. \tag{25}$$

The probability distribution for $q > 0$ is

$$p_0 = \ln\left[\frac{qe}{q + \bar{n}(1 - e^{-q})}\right] \qquad p_n = \frac{q^n}{n!}\sum_{k=1}^{\infty} k^{n-1}\left(\frac{\bar{n}\,e^{-q}}{q+\bar{n}}\right)^k \qquad n \geqslant 1. \tag{26}$$

Although all coefficients p_n with $n \geqslant 1$ are positive for any positive q and $\bar{n}$, the vacuum probability p_0 is nonnegative provided

$$\bar{n} \leqslant \frac{q(e-1)}{1 - e^{-q}}. \tag{27}$$

For $q \gg 1$ and $q \gg \bar{n}$ the probabilities (26) become close to (20):

$$p_0 \approx 1 - \bar{n}/q \qquad p_n \approx \frac{\bar{n}}{q}\left(\frac{q^n}{n!}\,\mathrm{e}^{-q}\right) \qquad n = 1, 2, 3, \ldots. \tag{28}$$

Using equations (16) and (17) we obtain Mandel's parameter q_- for the state $|\psi_-\rangle$ (4) in terms of the factorial moments of the initial arbitrary state $|\psi\rangle$

$$q_- = \frac{\bar{n}n^{(3)} - [n^{(2)}]^2}{\bar{n}n^{(2)}}. \tag{29}$$

The peculiar feature of the state generated by function (25) is that for this state $q_- = \bar{n}$, i.e., while for state $|\psi\rangle$, $\bar{n}$ is the mean photon number and q is Mandel's parameter, for state $|\psi_-\rangle$ the mean photon number is $\bar{n} + q$ and Mandel's parameter is $q_- = \bar{n}$. And the 'excess of photons' of state $\hat{a}|\psi_-\rangle$ is $\bar{n}$.

Going to the limit $q \to 0$ in (25) we obtain the generating function

$$G(z) = 1 - \ln[1 - \bar{n}(z - 1)] \tag{30}$$

which yields the following probabilities and factorial moments (16):

$$p_0 = 1 - \ln(1 + \bar{n}) \qquad p_n = \frac{1}{n}\left(\frac{\bar{n}}{1 + \bar{n}}\right)^n \qquad n \geqslant 1 \tag{31}$$

$$n^{(r)} = (r - 1)!\bar{n}^r \qquad r = 1, 2, \ldots. \tag{32}$$

The probability distribution with the same factorial moments (32) was found in [27] practically by the same method. The difference is that the authors of [27] calculated the generating function not for the probabilities, but for the factorial moments, having obtained, in particular, the function

$$Q_1(x) \equiv \sum_{\nu=0}^{\infty}(-x)^\nu\langle\hat{a}^{\dagger\nu}\hat{a}^\nu\rangle/\nu! = (A - 1)^{-1}\ln[(1 - A)\bar{n}x + 1] + 1 \tag{33}$$

where the *positive* constant A was considered as a measure of the relative deviation from the Poissonian case (the article [27] appeared before [9]):

$$[\langle\hat{a}^\dagger\hat{a}\rangle(\hat{a}^\dagger\hat{a}\rangle - 1)\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2]/\langle\hat{a}^\dagger\hat{a}\rangle^2 = -A. \tag{34}$$

In the limit case $A = 0$ formula (33) yields the factorial moments in the form (32) (but for $A \neq 0$ and $q = -A\bar{n} \neq 0$ the functions (25) and (33) give different values of factorial moments).

The statistics of the probability distribution (31) (not found explicitly in [27]) is quite different from the Poissonian statistics, although this distribution gives Mandel's factor $q = 0$ and the same first two factorial moments as in the coherent state with $|\alpha|^2 = \bar{n}$. The corresponding quantum state was named in [27] the *sub-coherent state* (the same name was used in [17] for quite different states possessing sub-Poissonian statistics). We can note in this connection that one of many possible pure states corresponding to the distribution (31) is a special case of the family of so-called *logarithmic states* introduced in [28]

$$|\psi\rangle_{\log} = c|0\rangle + \sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}}|n\rangle \qquad |c|^2 = 1 + \ln(1 - |z|^2) \qquad \overline{n} = \frac{|z|^2}{1 - |z|^2}. \tag{35}$$

5. A distinguished role of the exponential phase operator and phase coherent states

Acting on the 'logarithmic' state (35) by the operator $\hat{a}$ we arrive at its 'subtracted' partner

$$|\psi_-\rangle_{\log} = \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n|n\rangle \tag{36}$$

known under the names *coherent phase state* [29–31], *harmonious state* [32], or *pseudothermal state* [33] (see also [34]).

The mean number of quanta in the 'sub-coherent' state (35) cannot exceed a small value $e - 1 \approx 1.72$. However, it is easy to show that there exist many *sub-coherent states* ($q = 0$) with arbitrary values of $\bar{n}$. An example is the superposition of two Fock states (8). Considering for simplicity the case of $n \gg m$, one can verify that $q = 0$ either for $r \approx m/n^2$ (when $\bar{n} \approx m$) or for $1 - r \approx 1/n$ (when $\bar{n} \approx n$).

The coherent phase state (36) is the eigenstate of the *exponential phase operator*

$$\hat{E}_- \equiv \sum_{n=1}^{\infty} |n-1\rangle\langle n| = (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a} \equiv (\hat{n}+1)^{-1/2}\hat{a} \tag{37}$$

introduced in [35] and discussed in [30, 31, 36]. Its Hermitian conjugated partner is

$$\begin{aligned} &\hat{E}_+ = \sum_{n=1}^{\infty} |n\rangle\langle n-1| = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger)^{-1/2} \equiv \hat{a}^\dagger(\hat{n}+1)^{-1/2} \\ &[\hat{E}_-, \hat{E}_+] = |0\rangle\langle 0| \qquad \hat{E}_+\hat{E}_- = \hat{1} - |0\rangle\langle 0|. \end{aligned} \tag{38}$$

Acting on the Fock states, the operators $\hat{E}_\pm$ shift the number of quanta by ± 1 *without changing the amplitude* of the state vector:

$$\hat{E}_-|n\rangle = (1-\delta_{n0})|n-1\rangle \qquad \hat{E}_+|n\rangle = |n+1\rangle. \tag{39}$$

Therefore, applying the operators $\hat{E}_\pm$ to an arbitrary state $|\psi\rangle$ (3) we obtain the following normalized states $|\widetilde{\psi}_\pm\rangle \sim \hat{E}_\pm|\psi\rangle$:

$$|\widetilde{\psi}_-\rangle = \frac{1}{\sqrt{1-|c_0|^2}}\sum_{n=0}^{\infty} c_{n+1}|n\rangle \qquad |\widetilde{\psi}_+\rangle = \sum_{n=0}^{\infty} c_n|n+1\rangle. \tag{40}$$

Their quantum number generating functions are related to the initial generating function $G(z)$ (15) by simple relations

$$\widetilde{G}_-(z) = \frac{G(z)-G(0)}{z(1-p_0)} \qquad \widetilde{G}_+(z) = zG(z). \tag{41}$$

Consequently, the mean numbers of quanta in the new states (40) are connected with the mean number $\bar{n}$ in the initial state as follows:

$$\widetilde{N}_- \equiv \langle\widetilde{\psi}_-|\hat{n}|\widetilde{\psi}_-\rangle = \frac{\bar{n}}{1-p_0} - 1 \qquad \widetilde{N}_+ \equiv \langle\widetilde{\psi}_+|\hat{n}|\widetilde{\psi}_+\rangle = \bar{n} + 1. \tag{42}$$

We see that the operator $\hat{E}_+$ adds *exactly one* quantum to *any* quantum state, whereas the operator $\hat{E}_-$ *removes* exactly one quantum from any state *which has no contribution from the vacuum state* (i.e., if $p_0 = 0$). Therefore, perhaps, not operators $\hat{a}$ and $\hat{a}^\dagger$, but operators $\hat{E}_-$ and $\hat{E}_+$ should be named 'annihilation' and 'creation' operators in the literal meaning of these words. (See in this connection study [37], where it was shown how the states $\hat{E}_+^m|\psi\rangle$ could arise as a result of the interaction of Jaynes–Cummings type between atoms and cavity fields or between motional and internal degrees of freedom of trapped ions. *Shifted thermal states*, which can be written as $\hat{\rho}_{\text{th}}^{(\text{shift})} = \hat{E}_+^m\hat{\rho}_{\text{th}}\hat{E}_-^m$, have been considered in [38], whereas methods of generating such states in a micromaser were discussed in [39].)

For the coherent states $p_0 = \exp(-\bar{n})$, and for $\bar{n} \ll 1$ we have $\widetilde{N}_- = \bar{n}/2$. For the coherent phase states (36) we have $\widetilde{N}_- \equiv \bar{n}$ for any $\bar{n} = |z|^2/(1-|z|^2)$. For many other states with $p_0 > 0$, one can always make the 'modified photon excess'

$$\widetilde{q} \equiv \widetilde{N}_- - \bar{n} = \frac{p_0 \bar{n}}{1-p_0} - 1 \tag{43}$$

positive by increasing the value of $\bar{n}$, which is, in a generic case, a free parameter independent of p_0. This can easily be seen, e.g., in the example (20), when the quantity

$$(1-p_0)\widetilde{q} = \bar{n} - \frac{\bar{n}+1}{\gamma}(1 - \mathrm{e}^{-\gamma\bar{n}})$$

is obviously positive for sufficiently large $\bar{n}$ and $\gamma > 1$. The same is true for the negative binomial states (18) with $\mu < 1$ and ξ sufficiently close to 1. Moreover, if $\bar{n} > 2(1-p_0)/p_0$, then $\widetilde{N}_- > \widetilde{N}_+$.

Even using operators of the form $\hat{A} = f(\hat{n})\hat{a}$ we can find the states $|\psi\rangle$ for which state $\hat{A}|\psi\rangle$ has more quanta (on average) than the state $|\psi\rangle$, for *arbitrary function* $f(\hat{n})$. The simplest example is the 'two-Fock' state (8) with $m = 0$. In this case $\bar{n} = rn$. Any operator $f(\hat{n})\hat{a}$ transforms this superposition into the single Fock state $|n-1\rangle$, and obviously $(n-1) > rn$ if $(1-r) > 1/n$.

6. Conclusion

Concluding, we have shown that the standard 'annihilation' boson operator diminishes the mean number of quanta only for 'nonclassical' states with negative values of Mandel's parameter, whereas there are many states (all of them *superpositions* or mixtures of Fock states), with the peculiarity that when acted on by $\hat{a}$, they produce new states whose mean number of quanta is increased. Moreover, we have shown that in certain cases the 'annihilation' operator $\hat{a}$ can increase the mean number of quanta much more effectively than its 'creation' partner $\hat{a}^\dagger$. We emphasized an interpretation (discovered in [12, 13] but still not well known) of Mandel's q-factor as the 'photon excess' in the mean number of quanta arising as a result of the action of operator $\hat{a}$ on the given quantum state. Besides, we have shown that maybe the pair of Hermitian conjugated *exponential phase operators* could be the best candidates for being called 'annihilation–creation' operators, although the 'lowering' operator from this pair also cannot diminish the mean number of quanta by exactly one unit, for quantum states with nonzero vacuum component. This gives a hint that, perhaps, in the quantum photocount formalism of continuous measurements, such as, for instance, that proposed by Srinivas and Davies [11] one should use instead of $\hat{a}$ and $\hat{a}^\dagger$ the operators $\hat{E}_-$ and $\hat{E}_+$, which would not violate the property V of the cited work (namely, the condition of boundedness for the counting rate). However, this issue is outside the objective of this paper, and we shall report on it elsewhere.

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References

[1] Bohm A 1994 *Quantum Mechanics: Foundations and Applications* (Berlin: Springer) p 53
Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press) p 479

Meystre P and Sargent M III 1991 *Elements of Quantum Optics* (Berlin: Springer) p 94
Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press) p 5
Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer) p 11
[2] Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (London: McGraw-Hill) p 39
Cohen-Tannoudji C, Diu B and Laloë F 1977 *Quantum Mechanics* (New York: Wiley) p 494
Loudon R 1994 *The Quantum Theory of Light* (Oxford: Clarendon) p 129
Schiff L I 1968 *Quantum Mechanics* (London: McGraw-Hill) p 183
[3] Sakurai J J 1967 *Advanced Quantum Mechanics* (Reading, MA: Addison-Wesley)
[4] Dirac P A M 1987 *The Principles of Quantum Mechanics* 4th edn (Oxford: Oxford University Press)
[5] Fock V 1928 *Z. Phys.* **49** 339
[6] Dakna M, Knöll L and Welsch D-G 1998 *Eur. Phys. J.* D **3** 295
[7] Hong L 1999 *Phys. Lett.* A **264** 265
[8] Wang X G 2000 *Opt. Commun.* **178** 365
[9] Mandel L 1979 *Opt. Lett.* **4** 205
[10] Mattle K, Michler M, Weinfurter H, Zeilinger A and Zukowski M 1995 *Appl. Phys.* B **60** S111
Ban M 1996 *J. Mod. Opt.* **43** 1281
Lvovsky A I and Mlynek J 2002 *Phys. Rev. Lett.* **88** 250401
[11] Srinivas M D and Davies E B 1981 *Opt. Acta* **28** 981
[12] Ueda M, Imoto N and Ogawa T 1990 *Phys. Rev.* A **41** 3891
[13] Lee C T 1993 *Phys. Rev.* A **48** 2285
[14] Aharonov Y, Lerner E C, Huang H W and Knight J M 1973 *J. Math. Phys.* **14** 746
[15] Stoler D, Saleh B E A and Teich M C 1985 *Opt. Acta* **32** 345
Lee C T 1985 *Phys. Rev.* A **31** 1213
[16] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[17] Brif C and Ben-Aryeh Y 1994 *Quantum Opt.* **6** 391
[18] Dodonov V V, Malkin I A and Man'ko V I 1974 *Physica* **72** 597
[19] Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492
Zhang Z and Fan H 1992 *Phys. Lett.* A **165** 14
Kis Z, Adam P and Janszky J 1994 *Phys. Lett.* A **188** 16
Xin Z Z, Duan Y B, Zhang H M, Hirayama M and Matumoto K I 1996 *J. Phys. B: At. Mol. Opt. Phys.* **29** 4493
Dodonov V V, Korennoy Y A, Man'ko V I and Moukhin Y A 1996 *Quant. Semiclass. Opt.* **8** 413
Man'ko V I and Wünsche A 1997 *Quant. Semiclass. Opt.* **9** 381
Sixdeniers J M and Penson K A 2001 *J. Phys. A: Math. Gen.* **34** 2859
Quesne C 2001 *Phys. Lett.* A **288** 241
[20] McNeil K J and Walls D F 1975 *Phys. Lett.* A **51** 233
[21] Sotskii B A and Glazachev B I 1981 *Opt. Spectrosc.* **50** 582
[22] Dodonov V V, Man'ko O V and Man'ko V I 1994 *Phys. Rev.* A **49** 2993
[23] Lee C T 1995 *Phys. Rev.* A **52** 3374
[24] de Lima A F and Baseia B 1996 *Phys. Rev.* A **54** 4589
[25] Aharonov Y, Huang H W, Knight J M and Lerner E C 1971 *Lett. Nuovo Cimento* **2** 1317
[26] Joshi A and Lawande S V 1989 *Opt. Commun.* **70** 21
Matsuo K 1990 *Phys. Rev.* A **41** 519
[27] Baltes H P, Quattropani A and Schwendimann P 1979 *J. Phys. A: Math. Gen.* **12** L35
[28] Simon R and Satyanarayana M V 1988 *J. Mod. Opt.* **35** 719
[29] Lerner E C, Huang H W and Walters G E 1970 *J. Math. Phys.* **11** 1679
Ifantis E K 1972 *J. Math. Phys.* **13** 568
Shapiro J H and Shepard S R 1991 *Phys. Rev.* A **43** 3795
Chaturvedi S, Kapoor A K, Sandhya R, Srinivasan V and Simon R 1991 *Phys. Rev.* A **43** 4555
Vourdas A 1992 *Phys. Rev.* A **45** 1943
Hall M J W 1993 *J. Mod. Opt.* **40** 809
Brif C and Ben-Aryeh Y 1994 *Phys. Rev.* A **50** 3505
[30] Vourdas A, Brif C and Mann A 1996 *J. Phys. A: Math. Gen.* **29** 5887
[31] Wünsche A 2001 *J. Opt.* B **3** 206
[32] Sudarshan E C G 1993 *Int. J. Theor. Phys.* **32** 1069
[33] Dodonov V V and Mizrahi S S 1995 *Ann. Phys., NY* **237** 226
[34] Dodonov V V 2002 *J. Opt.* B **4** R1
[35] Susskind L and Glogower J 1964 *Physics* **1** 49
[36] Carruthers P and Nieto M 1968 *Rev. Mod. Phys.* **40** 411
Paul H 1974 *Fortschr. Phys.* **22** 657

Bergou J and Englert B-G 1991 *Ann. Phys., NY* **209** 479
Lukš A and Peřinová V 1994 *Quantum Opt.* **6** 125
Ban M 1995 *Phys. Lett.* A **199** 275
Lynch R 1995 *Phys. Rep.* **256** 367
Royer A 1996 *Phys. Rev.* A **53** 70
[37] Moya-Cessa H, Chavez-Cerda S and Vogel W 1999 *J. Mod. Opt.* **46** 1641
[38] Lee C T 1997 *Phys. Rev.* A **55** 4449
[39] Scully M O, Meyer G M and Walther H 1996 *Phys. Rev. Lett.* **76** 4144