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Tunnelling of narrow Gaussian packets through delta potentials

M A Andreata[1] **and V V Dodonov**[2]

[1] Departamento de Física, Universidade Federal de São Carlos, Via Washington Luis Km 235, São Carlos, 13565-905, SP, Brasil

[2] Instituto de Física, Universidade de Brasília, Caixa Postal 04455, 70910-900 Brasília, DF, Brasil

E-mail: pmastro@df.ufscar.br and vdodonov@fis.unb.br

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Abstract

We consider transmission and reflection of narrow Gaussian wave packets by delta potentials in the cases of constant and specific (inverse linear) time-dependent strength. Both transmitted and reflected packets exhibit some 'squeezing' in the momentum probability distributions. Several different definitions of the transmission time are introduced and compared.

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1. Introduction

It is well known that the stationary Schrodinger equation with the potential

$$V(x) = \mathcal{Z}\delta(x) \tag{1}$$

admits a simple complete set of exact explicit solutions [1]. For this reason, it has been frequently used to model different phenomena in various fields of quantum physics, in the limit cases when the detailed form of a concrete potential is not essential and results depend on its integral characteristics, e.g., a product of effective height by effective width. In particular, such an approximation can be successfully applied to the analysis of various tunnelling devices in solid state physics [2], where the experimental technique of 'delta doping' has been used for creating different 'delta layers' for more than two decades [3]. Applications of three-dimensional delta potentials to the problems of atomic physics (photoionization, photodetachment) have been reviewed in [4]. Such potentials are also widely used in the theory of Bose Einstein condensates [5, 6]. Exactly solvable models of many-body systems interacting via delta potentials were considered in [7, 8].

Various solutions of the *time-dependent* Schrodinger equation with potential (1) were also considered by many authors, beginning, perhaps, with the paper [9], where an original approach to the initial-value problem was proposed. In particular, many efforts have been direc ed oward nding he propaga or $G(x, x'; t)$ [10 15], which enable o calc la e he evol ion of an ini ial wavef nc ion $\psi(x, 0)$ according o he rela ion

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, x'; t)\psi(x', 0)\,\mathrm{d}x'. \tag{2}$$

Generali a ion o he ca e of a ime-dependen reng h $\mathcal{Z}(t)$ were con idered in [16 18]. Propaga or for *moving* del a po en ial were ob ained in [17, 19 21]. For o her die and generali a ion ee, e.g., [22 36].

However, concre e calc la ion of he in egral (2) and he anal i of evol ion of differen ini ial locali ed wave packe in he pre ence of del a- pe po en ial were performed onl in a few die [9, 13, 37]. The aim of o r paper i o con ider re ec ion and ran mi ion b he po en ial (1) (wi h con an or peci c ime-dependen reng h $\mathcal{Z}(t)$) of ini ial *narrow Gaussian* packe , wi h an empha i on *slowly moving* packe . Thi pecial ca e co ld be reali ed in e perimen wi h l racold a om . In par ic lar, i i clo el rela ed o he phenomenon of *quantum deflection* of low packe from re ec ing and emi ran paren mirror [38 40]. For o her recen p blica ion devo ed o propaga ion and re ec ion of q an m packe (ma er wave) ee, e.g., [41 48].

O r plan i a follow . In ec ion 2 we derive anal ical form lae de cribing re ec ion and ran mi ion of ini ial narrow Ga ian packe hro gh he a ionar del a barrier. We di c he effec of ' q ee ing' he momen m and coordina e probabili den i ie , he ime-dependen and a mp o ical ran mi ion probabili ie and he concep of condi ional average val e . In ec ion 3 we compare differen po ible de ni ion of he ' ran mi ion ime'. In ec ion 4 we con ider a generali a ion o he ca e of a peci c (inver e linear) ime dependence of he del a-po en ial reng h. Sec ion 5 con ain concl ion .

2. Evolution of packets in a stationary delta potential

The e plici form of he propaga or for he del a po en ial wa fo nd in everal paper [10, 12, 13]

$$G_{\mathcal{Z}}(x, x'; t) = (2\pi \mathrm{i}t)^{-1/2}\,\mathrm{e\;p}\left[\frac{\mathrm{i}(x-x')^2}{2t}\right] \\ -\frac{\mathcal{Z}}{2}\,\mathrm{e\;p}\left[\mathcal{Z}(|x|+|x'|)+\frac{\mathrm{i}\mathcal{Z}^2 t}{2}\right]\mathrm{erfc}\left[\frac{|x|+|x'|+\mathrm{i}\mathcal{Z}t}{\sqrt{2\mathrm{i}t}}\right] \tag{3}$$

where

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty}\mathrm{e\;p}(-y^2)\,\mathrm{d}y \equiv 1-\mathrm{erf}(z) \tag{4}$$

i he complemen ar error f nc ion [49, 50]. To implif form lae we a me formall ha $h = m = 1$. The re rn o he dimen ional variable can be performed b mean of he replacemen

$$t \to \frac{ht}{m} \qquad \mathcal{Z} \to \frac{m\mathcal{Z}}{h^2}. \tag{5}$$

For imaginar ime, i.e. for he eq ilibri m den i ma ri , a form la eq ivalen o (3) wa ob ained in [51], and he eq ilibri m Wigner f nc ion wa calc la ed in [52]. The econd erm in he righ -hand ide of (3) i ome ime called he *Moshinsky function* [13, 53, 54], beca e a imilar e pre ion appeared in he paper [55] devo ed o he 'diffrac ion in ime' problem. Eq a ion (3) hold for an ign of $\mathcal{Z}$ (al ho gh in ome paper i wa derived for a rac ive or rep l ive po en ial onl). O her (in egral) repre en a ion for he propaga or were given in [11, 14, 15]. For $\mathcal{Z} \to \infty$ and ed val e of x, x' and t, he propaga or (3) goe o he propaga or in he half- pace con ned wi h an impene rable wall (for $x, x' > 0$)

$$G_\infty(x, x'; t) = (2i\pi t)^{-1/2} \left\{ \exp\left[-\frac{(x'-x)^2}{2it}\right] - \exp\left[-\frac{(x'+x)^2}{2it}\right] \right\} \tag{6}$$

d e o he a mp o ical form la [49, 50]

$$\operatorname{erfc}(x) \approx \frac{\exp(-x^2)}{x\pi^{1/2}} \qquad |x| \to \infty \quad |\arg x| < \frac{3\pi}{4}. \tag{7}$$

E ac or q a i-cla ical propaga or in he pre ence of addi ional po en ial and for vario bo ndar condi ion on a half-line or in hree-dimen ional domain epara ed b creen and li were ob ained, e.g., in [56 63].

P ing e pre ion (3) in he in egral (2) one ho ld remember ha $G(x, x'; t)$ ha differen anal ical form for $x' > 0$ and $x' < 0$, o ha he in egra ion ho ld be performed epara el over he region $x' > 0$ and $x' < 0$ (and he re l i differen for $x > 0$ and $x < 0$). However, if we ppo e ha he ini ial wavef nc ion i well locali ed far awa o he righ from he origin, being eq al o ero for $x < 0$, hen he in egra ion in (2) ho ld be performed over $x' > 0$ onl , i.e. we ma replace $|x'|$ b x' i elf. We con ider in hi paper a Ga ian ini ial a e[3]

$$\psi(x, 0) = (\pi s^2)^{-1/4} \exp\left[-\frac{(x-x_c)^2}{2s^2} + ip_0 x\right] \qquad x_c \gg s. \tag{8}$$

The form (8) implie ha we con ider he a e wi ho ini ial correla ion be ween he coordina e and momen m (i wa hown in [39, 40] ha non ero ini ial correla ion coef cien doe no change e en iall he pic re of re ec ion or ran mi ion of narrow low packe from barrier). Al ho gh f nc ion (8) ha a non ero ' ail' in he region $x < 0$, hi ail i e ponen iall mall (nder he a med condi ion $x_c \gg s$) and doe no give an igni can con rib ion. Therefore, he in egra ion in (2) can be formall e ended o he whole a i $-\infty < x' < \infty$. U ing he in egral [64]

$$\int_{-\infty}^{\infty} dx \exp(-ax^2 + bx) \operatorname{erfc}(x) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \operatorname{erfc}\left(\frac{b}{2\sqrt{a(1+a)}}\right) \tag{9}$$

and in rod cing new dimen ionle variable and parame er

$$\tilde{x} = \frac{x}{x_c} \qquad \tilde{p} = \frac{ps}{h\sqrt{2}} \qquad \tau = \frac{th\sqrt{2}}{msx_c} \qquad \mathcal{B} = \frac{ms\mathcal{Z}}{h^2\sqrt{2}} \qquad \upsilon = 1 + i\beta\tau \qquad \beta = \frac{x_c}{s\sqrt{2}} \tag{10}$$

we ob ain

$$\begin{aligned} \psi(\tilde{x}, \tau) = \left(\frac{2\beta^2}{\pi\upsilon^2}\right)^{1/4} \Bigg\{ & \exp\left[-\frac{\beta^2}{\upsilon}(\tilde{x} - 1 - \tilde{p}_0\tau)^2 + i\beta\tilde{p}_0(2\tilde{x} - \tilde{p}_0\tau)\right] \\ & - \mathcal{B}\sqrt{\pi\upsilon} \exp\left[\mathcal{B}^2\upsilon + 2\beta\mathcal{B}(|\tilde{x}| + 1) - \tilde{p}_0^2 + 2i\tilde{p}_0(\beta + \mathcal{B})\right] \\ & \times \operatorname{erfc}\left[\frac{\beta}{\sqrt{\upsilon}}(|\tilde{x}| + 1) + \mathcal{B}\sqrt{\upsilon} + \frac{i\tilde{p}_0}{\sqrt{\upsilon}}\right] \Bigg\}. \end{aligned} \tag{11}$$

3

For $\mathcal{B} > 0$ (repulsive potential) and $\beta \gg 1$, the real part of the argument of the erfc-function on the right-hand side of equation (11) is large and positive for any value of τ, so we can simplify this expression using the asymptotical formula (7):

$$\psi(\xi,\tau) \approx \left(\frac{2\beta^2}{\pi\upsilon^2}\right)^{1/4}\left\{\exp\left[-\frac{\beta^2}{\upsilon}(\xi - 1 - p_0\tau)^2 + \mathrm{i}\beta p_0(2\xi - p_0\tau)\right]\right.$$
$$\left. - \frac{\mathcal{B}\upsilon}{\beta(|\xi| + 1) + \mathcal{B}\upsilon + \mathrm{i}p_0}\exp\left[-\frac{\beta^2}{\upsilon}(|\xi| + 1 + p_0\tau)^2 - \mathrm{i}\beta p_0(2|\xi| + p_0\tau)\right]\right\}. \quad (12)$$

2.1. Reflected packet

If $\mathcal{B} \to \infty$, the pre-exponential factor in the second line of equation (12) tends to 1 (for fixed ξ and τ). In this limit, for $\xi > 0$ the right-hand side of (12) becomes a superposition of a freely expanding Gaussian packet (with zero mean initial momentum) and a packet reflected by an ideal boundary [38], whereas it goes to zero for $\xi < 0$. The probability density $|\psi(\xi,\tau)|^2$ rapidly oscillates in the semispace $\xi > 0$ (if $\mathcal{B} \neq 0$) due to interference between the freely propagating part of the wave packet and the part reflected from the potential. The asymptotical (as $\tau \to \infty$) *momentum distribution* $|\varphi(p)|^2$ does not depend on τ, but it contains rapidly oscillating terms containing sine or cosine functions of the big argument βp. After averaging over these oscillations (which do not really affect measurable quantities [39, 40]) we obtain a smooth distribution, which has different forms for positive and negative values of momentum

$$\overline{\mathcal{P}}_{\mathrm{a}}^{(\pm)}(p) = \begin{cases} |\varphi_0(p)|^2 + |\varphi_0(-p)|^2|\chi(p)|^2 & p > 0 \\ |\varphi_0(p)|^2(1 - |\chi(p)|^2) & p < 0 \end{cases} \quad (13)$$

where

$$|\chi(p)|^2 = \frac{\mathcal{B}^2}{}$$

packe [38, 39]. Howe er, no iola ion of he ncer ain rela ion happen , beca e σ_x grow wi h ime in all ca e a t^2 (wi h a maller coef cien for $|\mathcal{B}| \gg 1$ han for $\mathcal{B} = 0$), o ha he prod c $\sigma_p\sigma_x$ al o increa e wi h ime nlimi edl . On he o her hand, he *invariant uncertainty product* $\sigma_p\sigma_x - \sigma_{px}^2$ a mp o icall end o ome con an al e (propor ional o he parame er β^2, i.e. m ch grea er han $h^2/4$) [39, 40]. The a mp o ical mean al e of he momen m in he ca e di c ed eq al $\langle \tilde{p}_\infty \rangle = 1/\sqrt{2\pi}$ (no e ha he de ni ion of he dimen ionle momen m $\tilde{p}$ in eq a ion (10) differ b he fac or $\sqrt{2}$ from he de ni ion adop ed in [38 40]).

2.2. Transmitted packet

Now le di c he proper ie of he *transmitted* packe . For $\tilde{x} < 0$ he arg men of bo h e ponen ial f nc ion in form la (12) are he ame, o ha i read

$$\psi(\tilde{x},\tau) \approx \left(\frac{2\beta^2}{\pi\upsilon^2}\right)^{1/4} \frac{\beta(1-\tilde{x}) + \mathrm{i}\tilde{p}_0}{\beta(1-\tilde{x}) + \mathcal{B}\upsilon + \mathrm{i}\tilde{p}_0} \exp\left[-\frac{\beta^2}{\upsilon}(\tilde{x} - 1 - \tilde{p}_0\tau)^2 + \mathrm{i}\beta\tilde{p}_0(2\tilde{x} - \tilde{p}_0\tau)\right]. \tag{16}$$

Taking β large eno gh, one can neglec $\tilde{p}_0$ in he pre-e ponen ial fac or, a well a he parame er $\mathcal{B}$ in he real par of i denomina or (remember ha $\upsilon = 1 + \mathrm{i}\beta t$ and $\tilde{x} < 0$). Therefore, he probabili di rib ion o he lef of he barrier doe no depend on β nder he condi ion

$$\beta \gg 1 \qquad \beta \gg |\tilde{p}_0| \qquad \beta \gg \mathcal{B} \qquad \beta\tau \gg 1 \tag{17}$$

(which impl ha $\mathcal{B}$ i ni e, i.e. he barrier i no o all re ec ing):

$$\mathcal{P}^{(-)}(\tilde{x},\tau) \equiv |\psi(\tilde{x},\tau)|^2 \approx \left(\frac{2}{\pi\tau^2}\right)^{1/2} \frac{(1-\tilde{x})^2}{(1-\tilde{x})^2 + \tau^2\mathcal{B}^2} \exp\left[-\frac{2}{\tau^2}(\tilde{x} - 1 - \tilde{p}_0\tau)^2\right]. \tag{18}$$

I i clear ha for $\tilde{p}_0 < 0$ and $|\tilde{p}_0|\tau \gg 1$, he ma im m of he di rib ion (18) i a ained a $\tilde{x} - 1 = \tilde{p}_0\tau$. In hi ca e one can replace he erm $\tilde{x} - 1$ in he pre-e ponen ial fac or b $\tilde{p}_0\tau$ a he poin near he ma im m. Con eq en l , he ran mi ed packe (in he en e of i probabili den i) a mp o icall i clo e o he freel e panding ini ial Ga ian packe , m l iplied b he ran mi ion coef cien

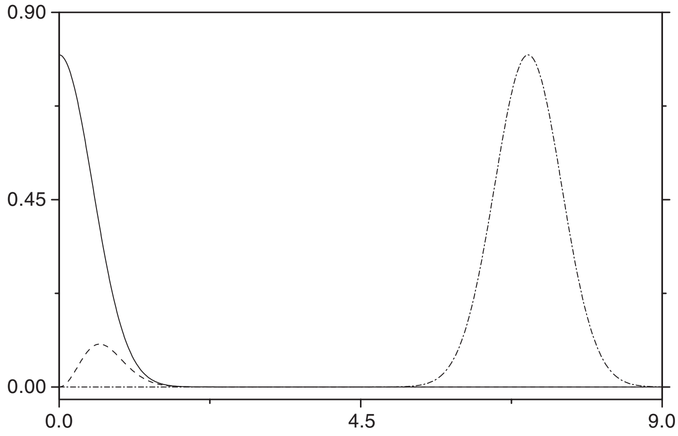
$$T(\tilde{p}) = 1 - |\chi(\tilde{p})|^2 = \frac{\tilde{p}^2}{\mathcal{B}^2 + \tilde{p}^2} \tag{19}$$

corre ponding o he ini ial momen m $\tilde{p}_0$. Thi wa no iced (wi ho e plici proof) a far back a in [9].

Howe er, hi imple re l hold onl if $|\tilde{p}_0|$ i no er mall. If $|\tilde{p}_0| \ll 1$, he i a ion i differen , and we con ider here he e reme ca e of $\tilde{p}_0 = 0$. In hi ca e he main par of he freel e panding ini ial packe reache he barrier a $\tau \sim 1$, herefore, he mo in ere ing regime i $\tau > 1$. In con ra o a freel e panding packe , who e ma im m i ed a he poin $\tilde{x} = 1$, he po i ion of he ma im m of he ran mi ed packe (18) wi h $\tilde{p}_0 = 0$ i grad all hif ed o he lef according o he form la

$$(1 - \tilde{x}_m)^2/\tau^2 = \tfrac{1}{2}\left(\mathcal{B}\sqrt{2 + \mathcal{B}^2} - \mathcal{B}^2\right). \tag{20}$$

In par ic lar, $|1 - \tilde{x}_m| \approx 2^{-1/4}\tau\sqrt{\mathcal{B}}$ for an almo ran paren barrier ($\mathcal{B} \ll 1$). On he o her hand, he eloci of he ma im m of he packe *does not depend* on $\mathcal{B}$ for an almo perfec l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0.45
0.00
0.0
4.5
9.0

Taking he limi $\tau \to \infty$ in eq a ion (24), we ob ain he a mp o ical ran mi ion probabili for he packe [40]

$$\mathcal{L}_\infty = \lim_{\tau\to\infty} \int_{-\infty}^{0} \mathcal{P}^{(-)}(x, \tau)\,\mathrm{d}x = \int_{-\infty}^{0} |\varphi_0(p)|^2 (1 - |\chi(p)|^2)\,\mathrm{d}p. \tag{25}$$

The 'condi ional' a mp o ical average val e of ome f nc ion of momen m can be de ned a [40, 65]

$$\langle\langle f(p)\rangle\rangle_\infty = \mathcal{L}_\infty^{-1} \int_{-\infty}^{0} f(p)|\varphi_0(p)|^2 (1 - |\chi(p)|^2)\mathrm{d}p. \tag{26}$$

The ph ical meaning of hi de ni ion eem o be clear: i corre pond o he a i ic of onl ho e even which are rela ed o he de ec ion of a par icle *behind the barrier*. For large nega ive val e of p_0, $\mathcal{L}_\infty = T(p_0)$, moreover, one can replace a lowl var ing re ec ion coef cien $|\chi(p)|^2$ b i val e a $p = p_0$ (keeping in mind ha he main con rib ion o he in egral in (26) i from a mall region near he poin p_0, d e o he e ponen ial form of he ini ial momen m di rib ion). Con eq en l , in hi a mp o ical ca e we have $\langle\langle f(p)\rangle\rangle_\infty \approx \langle f(p)\rangle_{t=0}$. In par ic lar, $\langle\langle p\rangle\rangle_\infty \approx p_0$.

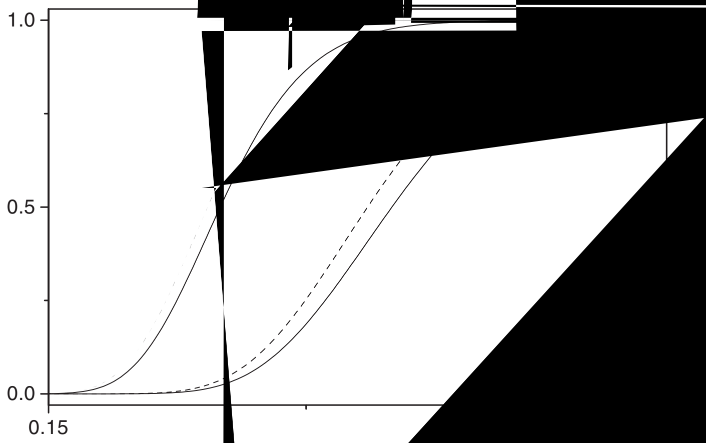
The i a ion i differen if $|p_0| < 1$. We con ider he ca e of $p_0 = 0$, when he effec i ma imal. The calc la ion are impli ed in he ca e of an almo perfec l re ec ing barrier wi h $\mathcal{B} \gg 1$, when he plane wave ran mi ion coef cien can be impli ed a $1 - |\chi(p)|^2 \approx p^2/\mathcal{B}^2$. Then eq a ion (15) and (25) ield $\mathcal{L}_\infty = (8\mathcal{B}^2)^{-1}$. Moreover, he condi ional a mp o ical val e of he average momen m in he ran mi ed packe doe no depend on $\mathcal{B}$ in hi limi ing ca e: $|\langle\langle p\rangle\rangle| = \sqrt{2/\pi}$. No e ha hi val e i wice a high a he mean momen m of re ec ed par icle and he condi ional mean val e of he momen m of par icle moving o he lef in he ab ence of an barrier (for he Ga ian packe). Al o, he val e $|\langle\langle p\rangle\rangle|$ i ligh l ($2/\sqrt{\pi}$ ime) grea er han he momen m corre ponding o he veloci (21) of he peak of he ran mi ed packe , d e o he ni e wid h of he ran mi ed packe in he momen m pace, which eq al

$$\overline{\Delta_p} \equiv \sqrt{\langle\langle p^2\rangle\rangle - \langle\langle p\rangle\rangle^2} = \sqrt{(3\pi - 8)/(4\pi)} \approx 1/3.$$

We ee ha $\overline{\Delta_p}$ i le han he wid h of he ini ial momen m di rib ion (15) (which eq al 1/2 for he dimen ionle variable p). Con eq en l , he ran mi ed packe al o e hibi ome ' q ee ing' in he momen m di rib ion. I become narrower han he ini ial one, beca e he low energ plane wave componen of he ini ial packe are ran mi ed hro gh he barrier wi h m ch maller probabili ie han he high energ one , o ha he low energ componen are 'lo ' in he ran mi ed packe . Thi al o e plain wh he condi ional average val e of he momen m in he ran mi ed packe i grea er han ha in he re ec ed packe . The prod c of condi ional ncer ain ie $\overline{\Delta_p}\,\overline{\Delta_x}$ linearl increa e wi h ime (in he a mp o ical regime), al ho gh i val e i le han ha in he ca e of a free packe .

3. Transmission times

There i a va li era re on he problem of ' nnelling ime' [66 68] or 'arrival ime' [69] of q an m par icle moving in vario po en ial or pa ing hro gh po en ial barrier . However, he ca e of nnelling hro gh he *delta barrier* eem o have been mi ed (he 'dela ime' for del a barrier wa con idered recen l ing he Floq e formali m [70] or ini ial c -off plane wave [71]). E plici e pre ion for he ime-dependen packe give a rare po ibili of d ing he problem in de ail in hi pecial ca e.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0.0
0.15

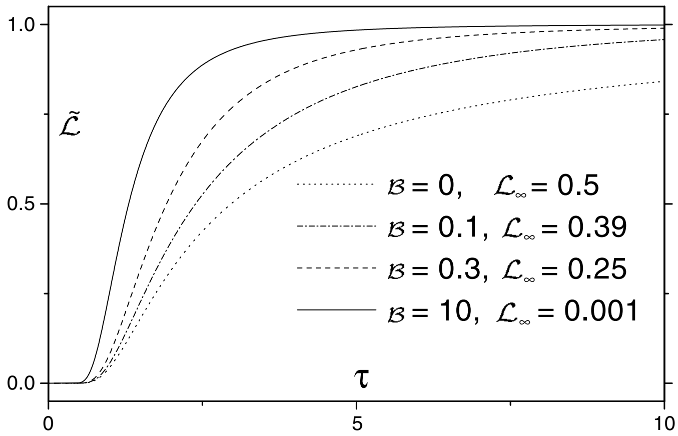


Figure 3. Normali ed ran mi ion probabili $\tilde{\mathcal{L}}(\tau) = \mathcal{L}(\tau)/\mathcal{L}_\infty$ for he ini ial packe wi h ero mean val e of he momen m $p_0 = 0$, for differen val e of he dimen ionle reng h of del a po en ial $\mathcal{B}$ (from bo om o op): 0; 0.1; 0.3; 10.

val e of variable τ which give $\xi = \pm 2$. Therefore, $\tau_{0.01} \approx 2\sqrt{2}/p_0^2$ (or $4mhx_c/(p_0^2 s)$ in he dimen ion variable). Thi val e doe no depend on he reng h of he del a po en ial (remember ha we con ider he limi of packe wi h large ini ial momen m p_0, e ceeding he wid h of he ini ial momen m di rib ion, which ha an order of h/s), being de ermined comple el b he difference be ween he in an of ime when he plane wave componen of he ini ial packe , corre ponding o he 'effec ive border ' $p = p_0 \pm 2h/s$ of he momen m di rib ion (15), reach he po i ion of he barrier $x = 0$ from he ini ial poin x_c.

In he ca e of ero ini ial momen m $p_0 = 0$, he in egral (27) can be calc la ed e ac l if $\mathcal{B} = 0$ and appro ima el if $|\mathcal{B}| \gg 1$ (when one can neglec he erm y^2 in he denomina or of he in egrand):

$$\tilde{\mathcal{L}}(\tau) = \mathrm{erfc}(\sqrt{2}/\tau) \qquad \mathcal{B} = 0 \tag{30}$$

$$\tilde{\mathcal{L}}(\tau) = \mathrm{erfc}\left(\frac{\sqrt{2}}{\tau}\right) + \frac{\sqrt{8}}{\sqrt{\pi}\,\tau}\exp\left(-\frac{2}{\tau^2}\right) \qquad \mathcal{B} \gg 1. \tag{31}$$

Th , we ob ain he following a mp o ical beha io r of he normali ed ran mi ion probabili a $\tau \gg 1$ in he e wo oppo i e limi ca e :

$$\tilde{\mathcal{L}}(\tau) = \begin{cases} 1 - \dfrac{8\sqrt{2}}{3\sqrt{\pi}\,\tau^3} + \mathrm{O}(\tau^{-5}) & \mathcal{B} \gg 1 \\ 1 - \dfrac{2\sqrt{2}}{\sqrt{\pi}\,\tau} + \mathrm{O}(\tau^{-3}) & \mathcal{B} \ll 1. \end{cases} \tag{32}$$

Re l of n merical calc la ion of he f nc ion $\tilde{\mathcal{L}}(\tau)$ for differen val e of $\mathcal{B}$ are hown in g re 3. We ee ha he ime of ran mi ion of he packe o he lef half- pace in he pre ence of a del a barrier i hor er han for a freel e panding packe wi h ero ini ial momen m, and i goe o ome ni e a mp o ical val e for $\mathcal{B} \gg 1$. Eq a ion (32) how ha for $p_0 = 0$, ro ghl peaking, $\tau_\epsilon \sim \epsilon^{-1}$ for $\mathcal{B} \ll 1$ and $\tau_\epsilon \sim \epsilon^{-1/3}$ for $\mathcal{B} \gg 1$.

An ob io di ad an age of he parame er τ_ϵ i an arbi rarine in he choice of ϵ. Thi ambig i can be remo ed if one no e ha he f nc ion $\tilde{\mathcal{L}}(\tau)$ i *monotonic*: ee g re 2

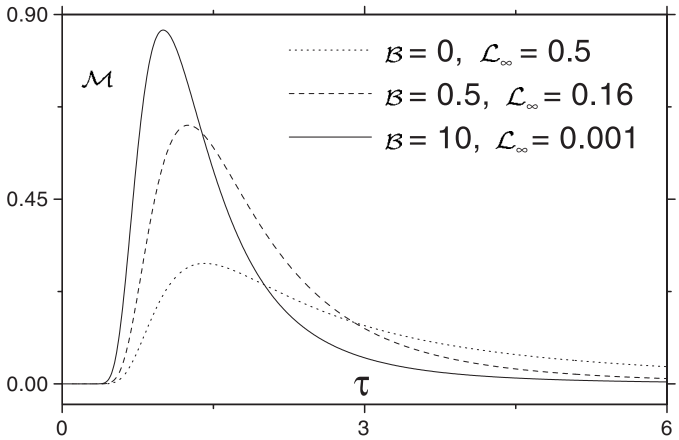


Figure 4. The ' ran mi ion ime probabili den i ' f nc ion (34) for $\mathcal{P}_0 = 0$ and differen val e of $\mathcal{B}$ (from bo om o op for τ mall and in he inver e order for τ big): 0; 0.5; 10.

and 3. Con eq en l , he deriva ive $\mathcal{M}(\tau) \equiv \mathrm{d}\mathcal{P}(\tau)/\mathrm{d}\tau$ i nonnega ive, and i can be con idered a he probabili den i of par icle ran mi ion hro gh he barrier in he in erval be ween τ and $\tau + \mathrm{d}\tau$, beca e $\int_0^\infty [\mathrm{d}\mathcal{P}(\tau)/\mathrm{d}\tau]\,\mathrm{d}\tau = 1$. Th , we can de ne he *mean transmission time* a (cf [72])

$$\tau = \int_0^\infty \tau \mathcal{M}(\tau)\,\mathrm{d}\tau. \tag{33}$$

In view of eq a ion (27) we have

$$\mathcal{M}(\tau) = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}\ \ \mathrm{p}[-2(1+\mathcal{P}_0\tau)^2/\tau^2]}{\mathcal{L}_\infty \tau^2 (1+\mathcal{B}^2\tau^2)}. \tag{34}$$

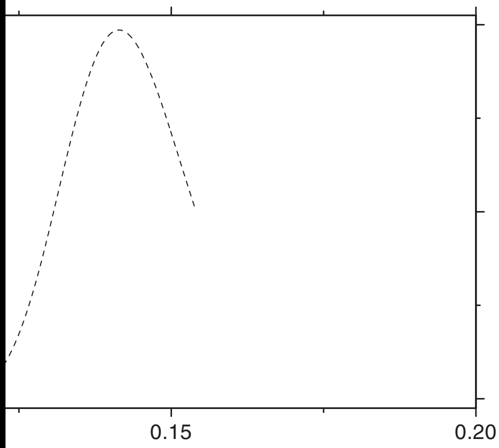
In he pecial ca e of $\mathcal{P}_0 = 0$, he in egral (33) wi h he f nc ion (34) can be red ced o he in egral e ponen ial f nc ion

$$E_1(x) \equiv \int_x^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t} \tag{35}$$

b mean of he b i ion $t = 2\mathcal{B}^2 + 2/\tau^2$, and we nd

$$\tau = \frac{\mathcal{L}_\infty^{-1}}{\sqrt{2\pi}} E_1(2\mathcal{B}^2)\,\mathrm{e}^{2\mathcal{B}^2}. \tag{36}$$

For $\mathcal{B} \gg 1$ one can e he a mp o ic of $E_1(x)$ for $x \gg 1$ or calc la e he in egral (33) direc l , neglec ing 1 wi h re pec o $\mathcal{B}^2\tau^2$ in he denomina or of (34) and making he b i ion $t = \tau^{-2}$ in he in egral. Bo h wa lead o he ame re l : ince $\mathcal{L}_\infty \sim \mathcal{B}^{-2}$, he mean ran mi ion ime goe o he con an a mp o ical val e $\tau_\infty = \sqrt{8/\pi}$ for $\mathcal{B} \gg 1$ and $\mathcal{P}_0 = 0$. However, he in egral (33) wi h he f nc ion (34) diverge a $\mathcal{B} \to 0$, o ha $\tau = \infty$ for he free packe wi h an val e of $\mathcal{P}_0$. Al ho gh one co ld r o nd ome e plana ion for ch behavio r in he ca e of $\mathcal{P}_0 = 0$, where he f nc ion $\mathcal{M}(\tau)$ behave more or le differen l for $\mathcal{B} = 0$ and $\mathcal{B} \gg 1$ (ee g re 4), here are no rea onable ph ical e plana ion for he divergence of ran mi ion ime a $\mathcal{B} \to 0$ for $|\mathcal{P}_0| \gg 1$, where he plo of $\mathcal{M}(\tau)$ prac icall do no depend on $\mathcal{B}$: ee g re 5. Thi divergence eem , herefore, o be a ma hema ical ar efac , indica ing ha he in egral (33) i no , in fac , a good mea re of he ran mi ion ime.


0.15
0.20

ol ion for $t > t_*$. A he cri ical in an t_*, one can e he a mp o ical form la (7) o ob ain he e pre ion re embling (6), b wi h ome modi ca ion :

$$G_*(x, x'; t_*) = (2\mathrm{i}\pi t_*)^{-1/2}\left\{\mathrm{e\ p}\left[\frac{\mathrm{i}(x'-x)^2}{2t_*}\right] - \left(1 - \frac{\mathrm{i}|x|}{\mathcal{Z}t_*}\right)^{-1}\mathrm{e\ p}\left[\frac{\mathrm{i}(|x'|+|x|)^2}{2t_*}\right]\right\}. \quad (40)$$

No e ha he modif ing pre-e ponen ial fac or doe no depend on he econd arg men x' of he propaga or (o er which he in egra ion in form la (2) i performed).

Appl ing he propaga or (39) o he ini ial a e (8), we ob ain he generali a ion of form la (11)

$$\psi(\xi, \tau) = \left(\frac{2\beta^2}{\pi\upsilon^2}\right)^{1/4}\mathrm{e\ p}\left[-\frac{\beta^2}{\upsilon}(\xi - 1 - p_0\tau)^2 + \mathrm{i}\beta p_0(2\xi - p_0\tau)\right]$$
$$- \frac{\mathcal{B}\left(2\pi\beta^2\right)^{1/4}}{[\zeta(1+\mathrm{i}A)]^{1/2}}\,\mathrm{erfc}\left[\frac{\beta(1+\mathrm{i}A)|\xi| + \mathcal{B}\upsilon + \zeta(\beta + \mathrm{i}p_0)}{[\zeta(1+\mathrm{i}A)\upsilon]^{1/2}}\right]$$
$$\times\,\mathrm{e\ p}\left[\frac{\beta|\xi|}{\zeta}(2\mathcal{B} + \mathrm{i}A\beta|\xi|) + \frac{\mathcal{B}^2\upsilon}{\zeta(1+\mathrm{i}A)} + \frac{2\mathrm{i}p_0(\beta+\mathcal{B}) + \beta(2\mathcal{B} - \mathrm{i}A) - p_0^2}{1+\mathrm{i}A}\right] \quad (41)$$

where

$$\zeta(\tau) = 1 + A\beta\tau \qquad A = \alpha m s^2/h \quad (42)$$

and o her ariable and parame er were de ned in (10). Ob io l , he parame er α^{-1} ha he meaning of ime of e i ence of he barrier (for $\alpha > 0$), wherea he q an i ms^2/h charac eri e he ime of preading of he ini ial packe . Therefore, we con ne o r el e o he anal i of he ca e $|A| \ll 1$ (o herwi e he del a po en ial prac icall di appear long before he main par of he packe reache he poin $x = 0$). On he o her hand, he prod c $A\beta \sim \alpha m s x_c/h$ ha he meaning of he ra io of he ime nece ar o reach he po i ion of he barrier (x_c/v_s, where he preading eloci ha he order h/ms) o he ime of e i ence of he barrier, o ha i i rea onable o ppo e ha $|A|\beta$ can be of he order of ni or bigger (o herwi e we ha e he ca e of prac icall a ionar del a po en ial). We hall al o a me ha

$$\beta \gg 1 \qquad \beta\tau \gg 1 \qquad \beta \gg |p_0| \qquad \beta \gg |\mathcal{B}|. \quad (43)$$

Le con ider r he ca e of α po i i e. Replacing again he complemen ar error f nc ion in (41) b i a mp o ical form (7) and aking in o acco n he re ric ion (43), one can erif ha he probabili den i of he ran mi ed par of he packe (for $\xi < 0$) can be e pre ed in almo he ame form a in eq a ion (18), wi h he onl difference ha he erm $1 - \xi$ ho ld be replaced b $\zeta(\tau) - \xi$ *in the pre-exponential factor* (b no in he arg men of he e ponen ial):

$$\mathcal{P}^{(-)}(\xi, \tau) \approx \left(\frac{2}{\pi\tau^2}\right)^{1/2}\frac{(\zeta - \xi)^2}{(\zeta - \xi)^2 + \tau^2\mathcal{B}^2}\,\mathrm{e\ p}\left[-\frac{2}{\tau^2}(\xi - 1 - p_0\tau)^2\right]. \quad (44)$$

A a con eq ence, we ha e he following generali a ion of form la (34) for he ran mi ion ime probabili den i :

$$\mathcal{M}(\tau) = \mathcal{L}_\infty^{-1}\left(\frac{2}{\pi}\right)^{1/2}\frac{\zeta^2}{\tau^2(\zeta^2 + \mathcal{B}^2\tau^2)}\,\mathrm{e\ p}\left[-\frac{2}{\tau^2}(1 + p_0\tau)^2\right] \quad (45)$$

where

$$\mathcal{L}_\infty = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \mathrm{d}y\,\frac{(y + A\beta)^2}{(y+A\beta)^2 + \mathcal{B}^2}\,\mathrm{e\ p}[-2(y + p_0)^2]. \quad (46)$$

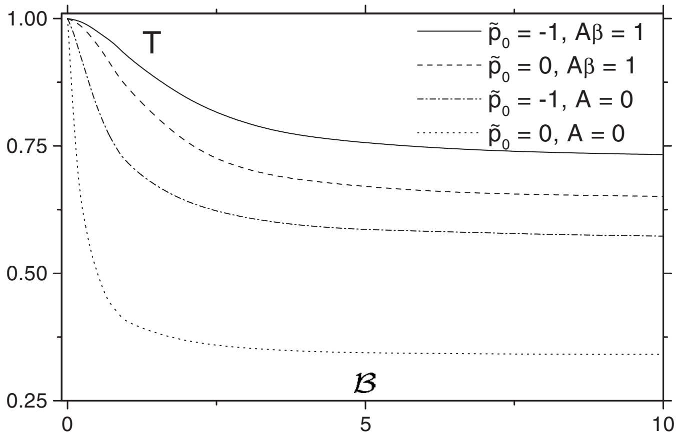


Figure 6. The ra io $T(\mathcal{B}) \equiv T_r(\mathcal{B})/T_r(0)$ a a f nc ion of $\mathcal{B}$ for ed val e of he mean ini ial momen m $\tilde{p}_0$.

o reach he barrier. B a hi momen in he non a ionar ca e he packe will 'mee ' no he ini ial barrier, b a barrier who e reng h became $A\beta$ ime weaker, d e o he peci c ime dependence (38).

5. Conclusion

We have anal ed he problem of ran mi ion and re ec ion of ini iall narrow Ga ian packe b he del a po en ial of con an and pecial ime-dependen reng h . We have ob ained appro ima e anal ical e pre ion for he coordina e and momen m probabili den i ie behind he barrier. The e e pre ion demon ra e ome ' q ee ing' bo h in he momen m and coordina e di rib ion (a oon a he e di rib ion reprod ce each o her in he a mp o ical regime af er ome caling of variable), compared wi h he ca e of free e pan ion in he ab ence of a barrier. None hele , he ncer ain rela ion are no viola ed, beca e he coordina e variance nlimi edl grow wi h ime (al ho gh lower han in he ca e of a free packe).

Al o, we have in rod ced wo f nc ion , which can be in erpre ed a a ran mi ion ime probabili den i and a o al ime-dependen ran mi ion probabili . U ing he e f nc ion , we have con idered everal po ible de ni ion of he ran mi ion ime and anal ed heir dependence on he parame er of he ini ial packe and he po en ial. Q ali a ivel , all he de ni ion lead o he concl ion ha he ran mi ion ime dimini he wi h increa e of he reng h of he del a po en ial. However, hi effec i igni can onl for low ini ial packe , being prac icall nob ervable in he ca e of packe wi h large nega ive ini ial average momen a. Moreover, he ran mi ion ime do no red ce nlimi edl , b he end o ome ni e a mp o ical val e a he reng h of he del a po en ial goe o in ni (when he o al ran mi ion probabili end o ero). The be de ni ion of he ran mi ion ime in he ca e nder d eem o be he inver e heigh of he ma im m of he ran mi ion ime probabili den i .

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