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LETTER TO THE EDITOR

The Wigner function associated with the Rogers-Szegö polynomials

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Abstract
A Wigner function associated with the Rogers-Szegö polynomials is proposed and its properties are discussed. It is shown that from such a Wigner function it is possible to obtain well-behaved probability distribution functions for both angle and action variables, defined on the compact support $-\pi \leqslant \theta < \pi$, and for $m \geqslant 0$, respectively. The width of the angle probability density is governed by the free parameter q characterizing the polynomials.

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Though the properties of the Wigner function associated with the harmonic oscillator, and therefore with the Hermite polynomials, are quite well known, the same cannot be said concerning the Rogers-Szegö polynomials (RSP) [1–6]

$$H_n(y) \equiv H_n(y;q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} y^r, \tag{1}$$

with the q-binomial

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^r)(1-q)\cdots(1-q^{n-r})} = \frac{(q)_n}{(q)_r(q)_{n-r}} \tag{2}$$

for r and n integers, $0 \leqslant r \leqslant n$ and $(0)_n = 1$. So the aim of the present letter is to propose a Wigner function related to the RSP and discuss the associated probability distribution functions extracted from it.

We recall that the Weyl–Wigner transformation associated with the well-known translational degree of freedom for Cartesian variables and moments of a particle has long been established and widely discussed in the literature [7–12]. On the other hand, in that context, the rotational degree of freedom has been scarcely touched upon. The treatments of this case were directly inferred from the previous one by means of symmetry arguments [13, 14], by the continuous limit of finite-dimensional Weyl–Wigner mappings [15, 16], or by the implementation of the appropriate kinematics relations [17]. It is clear in all these cases that one is dealing with functions of angular variables that have period 2π and the measure of this function space is simply unity.

It was shown that the RSP are associated with a realization of the q-deformed harmonic oscillator algebra [18–21], and are characterized by a discrete *positive* variable n and a continuous angle variable θ (an action–angle pair in contrast to the Hermite action–position) depending on a deformation parameter q. We are then able to write a Weyl–Wigner transformation from which we extract angle and action probability distribution functions. As such, we propose that the RSP can be used as good functions to describe phase states.

The RSP can be made periodic, with period 2π, and orthonormalized on the circle provided we first perform a proper choice for the variable y, $y = -q^{-1/2}\,\mathrm{e}^{\mathrm{i}\varphi}$, such that

$$H_n(y;q) = H_n(-q^{-1/2}\,\mathrm{e}^{\mathrm{i}\varphi};q) \tag{3}$$

and make use of the Jacobi ϑ_3-function [22]

$$\vartheta_3(\varphi;q) = \sum_{m=-\infty}^{\infty} q^{m^2/2}\,\mathrm{e}^{\mathrm{i}m\varphi} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mu m^2+\mathrm{i}m\varphi}, \tag{4}$$

($\mu = -(\ln q)/2$) as a measure function, in the same way as the Gaussian function is a measure function for the standard Hermite polynomials associated with the one-dimensional harmonic oscillator (note that $0 \leqslant q \leqslant 1$ implies $0 \leqslant \mu \leqslant \infty$).

Due to its properties, the Jacobi ϑ_3-function has already been proposed as a valuable function to describe particular limiting situations in quantum optics [23], and also as a coherent state for a particle on a circle where the angular variable now plays an essential role [24, 25]. In this case, the algebra is given in terms of the angular momentum and a unitary operator so that the commutation relation is $[J, U] = U$, U is a unitary operator associated with angular momentum shifts. Such a commutation relation was discussed long ago in the literature [26, 27], and was also obtained as the limiting case in finite-dimensional phase space representation of quantum mechanics [15]. It is worth noting that the Jacobi ϑ_3-function with integer argument was also proposed as a coherent state for the case of any finite-dimensional degrees of freedom [28], since in these cases the eigenvalue problem associated with the discrete Fourier matrix in the discrete basis [29] gives a solution which is directly expressed in terms of that Jacobi function. In this sense we see that the Jacobi ϑ_3-function plays a wider role in connection with coherent states and, in particular, with the rotational or action–angle degrees of freedom.

We also give what is sometimes known as the Rogers-Szegö function (RSF)

$$R_n(\varphi;q) = \frac{q^{n/2}}{[(q,q)_n]^{1/2}} H_n(-q^{-1/2}\,\mathrm{e}^{\mathrm{i}\varphi};q). \tag{5}$$

where [5, 30]

$$(x;q)_n \equiv (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}) = \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)} x^j. \tag{6}$$

The orthonormalization integral is written as

$$I_{mn}(q) = \int_{-\pi}^{\pi} H_m(-q^{-1/2}\,\mathrm{e}^{\mathrm{i}\varphi};q) H_n(-q^{-1/2}\,\mathrm{e}^{-\mathrm{i}\varphi};q)\vartheta_3(\varphi;q)\frac{\mathrm{d}\varphi}{2\pi}$$

and using the definition (1) we get

$$I_{mn}(q) = \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{r(r-1)/2} q^{s(s-1)/2} q^{-rs} = \frac{(q,q)_n}{q^n}\delta_{m,n}, \qquad (7)$$

a result discussed by Carlitz [4] (see the appendix for a proof).

In the same form as for the harmonic oscillator on the line, where the probability distribution for x is $\Phi^{(n)}(x;b)\,\mathrm{d}x = |H_n(x;b)|^2 \exp(-x^2/b^2)\,\mathrm{d}x$, with $H_n(x;b)$ the standard Hermite polynomial and b the harmonic oscillator width, acting as a controlling parameter, we may guess that the expression constructed with the ϑ_3-function and the RSF $\Omega^{(n)}(\varphi;q)\,\mathrm{d}\varphi = |R_n(\varphi;q)|^2 \vartheta_3(\varphi;q)\,\mathrm{d}\varphi$ is, as a matter of fact, a good candidate for the angle probability distribution, with q (or μ) a parameter controlling the distribution width.

Noting that $\vartheta_3(\varphi;q)$ is an even function of φ, and also guided by previous results [13–15], we propose a Weyl–Wigner mapping of an operator $\widehat{O}$ by taking the Fourier transform, namely,

$$O(m,\theta) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}m\widetilde{\theta}} \left\langle \theta - \frac{\widetilde{\theta}}{2} \right| \widehat{O} \left| \theta + \frac{\widetilde{\theta}}{2} \right\rangle \vartheta_3$$

On the other hand, by performing the summation over m in equation (9) we get the angle probability distribution, namely,

$$\Omega^{(n)}(\theta,\mu) = \sum_{m=-\infty}^{\infty} O^{(n)}(m,\theta) = \frac{q^n}{(q,q)_n}\sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mu t^2+\mathrm{i}t\theta}\sum_{r,s=0}^{n}(-1)^{r+s} \times \begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} \mathrm{e}^{\mu(r+s)}\,\mathrm{e}^{\mathrm{i}\theta(r-s)} \sum_{m=-\infty}^{\infty} \frac{\sin\left(m-\frac{t+r+s}{2}\right)\pi}{\left(m-\frac{t+r+s}{2}\right)\pi},$$

and since the sum over m equals 1 for $(t+r+s)/2$ integer or half-integer, we get

$$\Omega^{(n)}(\theta,\mu) = \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mu t^2+\mathrm{i}t\theta}\left\{\frac{q^n}{(q,q)_n}\sum_{r,s=0}^{n}(-1)^{r+s}\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix}\mathrm{e}^{\mu(r+s)}\,\mathrm{e}^{\mathrm{i}\theta(r-s)}\right\}.$$

The curly bracket can be immediately identified as

$$\frac{q^n}{(q,q)_n}\sum_{r,s=0}^{n}(-1)^{r+s}\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix}\mathrm{e}^{\mu(r+s)}\,\mathrm{e}^{\mathrm{i}\theta(r-s)} = |R_n(\theta;\mu)|^2,$$

so that, as anticipated, the angle probability distribution reads

$$\Omega^{(n)}(\theta,\mu) = \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mu t^2+\mathrm{i}t\theta}|R_n(\theta;\mu)|^2 = \vartheta_3(\varphi;\mu)|R_n(\theta;\mu)|^2, \qquad (12)$$

which is a well-behaved function in the compact support $-\pi \leqslant \theta < \pi$.

We can verify that the Wigner function is normalized to unity by just integrating equation (12) over its range of definition and recalling the orthogonalization procedure, or by summing expression (11) over m in the range $0 \leqslant m < \infty$.

As a first case of study it is now direct to particularize the Wigner function to the lowest Rogers-Szegö function, namely, $n = 0$, the vacuum state projector. In this case

$$O_0(m,\theta) = \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mu t^2+\mathrm{i}t\theta}\frac{\sin\left(m-\frac{t}{2}\right)\pi}{\left(m-\frac{t}{2}\right)\pi}, \qquad (13)$$

that gives $\Lambda^{(0)}(m) = \delta_{m,0}$ for the action probability distribution and the angle probability distribution simplifies to

$$\Omega^{(0)}(\theta,\mu) = \vartheta_3(\theta;\mu), \qquad (14)$$

since from equation (5) $R_0(\theta;\mu) = 1$.

In the same form, the normalized angle probability distribution for the second polynomial (projector $\widehat{O}_1$) is

$$\Omega^{(1)}(\theta,\mu) = \frac{\mathrm{e}^{-2\mu}}{1-\mathrm{e}^{-2\mu}}(1 - 2\,\mathrm{e}^{\mu}\cos\theta + \mathrm{e}^{2\mu})\vartheta_3(\theta;\mu).$$

Finally, it is worth noting that the angle probability distribution is μ-dependent as expected, so that the width of $\Omega^{(n)}(\theta,\mu)$ is governed by the free parameter q (or μ), which is the parameter of the deformed Heisenberg algebra [18, 31].

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Appendix. Carlitz orthogonality proof of the Rogers-Szegö polynomials

Let us first consider

$$I_{mn} = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{r}{2}(r-1)} q^{\frac{s}{2}(s-1)} q^{-rs}, \tag{A.1}$$

and using (6) in (A.1) we obtain

$$I_{mn} = \sum_{r=0}^{m} (-1)^{r} \begin{bmatrix} m \\ r \end{bmatrix} q^{\frac{r}{2}(r-1)} \prod_{s=0}^{n-1} (1 - q^{s-r}). \tag{A.2}$$

Now, without any loss of generality, we can assume that $m \leqslant n$ (the inverse could also be considered). There are two situations to be discussed. First, for $m < n$, it is evident that the product on the rhs of (A.2) will vanish for all r (the rhs is constituted of a sum of products. Each summand has a product of terms where one of them will give $(1 - q^{r-r}) = 0$, since, as $m < n, s$ will necessarily assume the value r). Therefore, the sum only has vanishing summands, since there will always be a zero factor in the products. Second, for $m = n$ there will be only one term to be considered, namely $r = m$. Thus

$$I_{mn} = (-1)^{n} q^{\frac{n}{2}(n-1)} \prod_{s=0}^{n-1} (1 - q^{s-n}) \delta_{m,n} = q^{-n} (q; q)_{n} \delta_{n,m}. \tag{A.3}$$

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