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SQUEEZING EXCHANGE AND ENTANGLEMENT BETWEEN RESONANTLY COUPLED MODES

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Abstract

We study the problem of squeezing exchange between two modes of the electromagnetic field modeled by quantum oscillators for the most general weak bilinear resonance coupling. Also we introduce a new measure of entanglement based on the cross covariances of the quadrature components of interacting modes. We compare the covariance measure with the measures based on the von Neumann and linear entropies of the subsystems, studying their dependences on time, coupling constants, and the initial state in the cases of parametric amplification and parametric conversion. In particular, we show that coherent states remain disentagled for all times and for any choice of coupling constants in the case of parametric converter (with accuracy up to second-order terms with respect to the strength of coupling). Also, we demonstrate that no bilinear coupling can squeeze the initial coherent state or improve the squeezing of the initial squeezed state in the case of a parametric amplifier. A strong sensitivity of the character of evolution to the choice of the set of coupling constants is discovered in the case of a parametric converter.

Keywords: coupled modes, resonance, parametric amplifier, parametric converter, general bilinear coupling, invariant squeezing coefficient, measure of entanglement, entropy, linear entropy, covariance matrix.

1. Introduction

The model of two coupled time-dependent harmonic oscillators has been studied extensively, being applied to various problems of quantum mechanics and quantum optics. For instance, it was used to describe quantum amplifiers [1–3] and converters [1, 4, 5]. Explicit exact solutions and propagators of the Schrödinger equation as well as solutions of the Heisenberg equations of motion were considered and applied to different problems in [6–15]. Squeezing, photon statistics, and entanglement in the system of two coupled oscillators were studied in [16–25]. As a rule, only specific couplings (mainly, through coordinates) were considered in all those papers. The most general bilinear coupling with time-independent coefficients was studied in detail in [26].

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The aim of our paper is to study the possibility of squeezing exchange between resonance-coupled modes with different (in the general case) frequencies, when coupling coefficients must be time dependent. Although this problem was already touched on in several publications [27–32], only specific couplings between the modes were considered before, whereas our goal is to consider the most general bilinear coupling. A more general problem of the quantum state exchange was studied recently in [33–35], but the specific problems related to the exchange of squeezing were not addressed there.

Another topic addressed in our paper is related to the quantitative characteristics of entanglement between coupled modes. We introduce a simple parameter based on the covariances of the quadrature components and study the evolution of this parameter for different types of coupling and different initial conditions, comparing it with the "traditional" entropic measures (based on the von Neumann and linear entropies of the subsystems).

The plan of the paper is as follows. In Sec. 2, we briefly discuss the general method of treating quantum systems with quadratic Hamiltonians, which is essentially reduced to finding some symplectic matrix responsible for the evolution of the system. In Secs. 3 and 4, respectively, we introduce the squeezing and entanglement coefficients expressed in terms of some invariant combinations of the elements of the covariance matrix. The evolution of these coefficients with time is studied in Sec. 5 where we consider separately the cases of a parametric amplifier and a parametric converter. A discussion of the results is presented in Sec. 6. The details of calculations are given in Appendices A and B.

2. Evolution of the Covariance Matrix of a Multidimensional Quantum System with a Quadratic Hamiltonian

We consider two coupled modes described by the total Hamiltonian

$$H = \frac{\hbar\omega_1}{2}\left(p_1^2 + x_1^2\right) + \frac{\hbar\omega_2}{2}\left(p_2^2 + x_2^2\right) + \hbar\varpi\Big(\gamma_1(t)p_1p_2 + \gamma_2(t)p_1x_2 + \gamma_3(t)x_1p_2 + \gamma_4(t)x_1x_2\Big), \tag{1}$$

where p_i and x_i are dimensionless quadrature-component variables, whose operators obey the commutation relations

$$[\hat{x}_j, \hat{p}_k] = i\delta_{jk}, \qquad \varpi \equiv \sqrt{\omega_1\omega_2}\,.$$

The dimensionless coupling constants

$$\gamma_i(t) = \gamma_i \cos(\eta t - \phi_i)$$

vary with time periodically with the same frequency η but the phases ϕ_i may be arbitrary. In the case of material oscillators with masses m_j, the dimension variables P_i and X_i are related to their dimensionless partners as follows:

$$p_i = \frac{P_i}{\sqrt{m_i\omega_i\hbar}}\,, \qquad x_i = X_i\sqrt{\frac{m_i\omega_i}{\hbar}}\,.$$

Hereafter, we assume formally $\hbar = 1$.

The Hamiltonian (1) is a special case of generic quadratic Hamiltonians, which can be written as follows (for the sake of simplicity, we confine ourselves to the homogeneous Hamiltonians):

$$H = \frac{1}{2}\mathbf{q}\mathcal{B}\mathbf{q}, \qquad \mathbf{q} \equiv (\mathbf{p}, \mathbf{x}), \tag{2}$$

where $\mathbf{x}$ is the N-dimensional "coordinate" vector and $\mathbf{p}$ is the canonically conjugated "momentum" vector; $\mathcal{B}(t)$ is a symmetric $2N\times 2N$ matrix

$$\mathcal{B} = \left\| \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right\|$$

consisting of four $N\times N$ blocks satisfying the conditions:

$$b_1 = \tilde{b}_1, \qquad b_4 = \tilde{b}_4, \qquad b_2 = \tilde{b}_3$$

(a tilde denotes matrix transposition). A remarkable property of quadratic Hamiltonians such as (2) is that they result in a linear dependence between the "initial" and "current" Heisenberg operators

$$\hat{q}_\mu(t) = \mathcal{L}_{\mu\alpha}(t)\hat{q}_\alpha(0), \tag{3}$$

where the coefficients $\mathcal{L}_{\mu\alpha}$ are the elements of certain symplectic matrix $\mathcal{L} \equiv \|\mathcal{L}_{\mu\alpha}\|$.

Introducing the inverse matrix $\Lambda = \mathcal{L}^{-1}$, we can rewrite Eq. (3) as follows:

$$\hat{\mathbf{q}}_0(t) = \Lambda(t)\hat{\mathbf{q}}. \tag{4}$$

Then $\hat{\mathbf{q}}_0(t)$ is the operator integral of motion in the Schrödinger picture [36, 37]. The blocks of Λ-matrix

$$\Lambda(t) = \left\| \begin{array}{cc} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{array} \right\|$$

determine the propagator of the Schrödinger equation with the Hamiltonian (2) in the coordinate representation [36, 37]:

$$G\left(\mathbf{x}_2;\mathbf{x}_1;t\right) = \Big[\det\left(-2\pi i\lambda_3\right)\Big]^{-1/2} \exp\left\{-\frac{i}{2}\left[\mathbf{x}_2\lambda_3^{-1}\lambda_4\mathbf{x}_2 - 2\mathbf{x}_2\lambda_3^{-1}\mathbf{x}_1 + \mathbf{x}_1\lambda_1\lambda_3^{-1}\mathbf{x}_1\right]\right\}. \tag{5}$$

To obtain the explicit time dependence, one has to solve the matrix equation

$$\dot{\Lambda} = \Lambda\Sigma\mathcal{B}, \qquad \Sigma = \left\| \begin{array}{cc} 0 & I_N \\ -I_N & 0 \end{array} \right\|, \qquad \Lambda(0) = I_{2N}, \tag{6}$$

where I_N denotes the $N\times N$ unity matrix.

Since Eq. (3) is linear, the "initial" and "current" values of the quadrature operators are related as

$$\langle q_\mu\rangle(t) = \mathcal{L}_{\mu\alpha}(t)\langle q_\alpha\rangle(0), \tag{7}$$

and this relation does not depend on the representation. Analogously, the second-order moments are given by

$$\langle q_\mu q_\nu\rangle(t) = \mathcal{L}_{\mu\alpha}(t)\langle q_\alpha q_\beta\rangle_0 \mathcal{L}_{\beta\nu}(t), \tag{8}$$

and the covariance matrix $\mathcal{M} = \|\mathcal{M}_{\mu\nu}\|$ evolves as follows:

$$\mathcal{M}(t) = \mathcal{L}(t)\mathcal{M}(0)\widetilde{\mathcal{L}}(t). \tag{9}$$

We use the following equivalent notation for the centralized second-order statistical moments:

$$\begin{aligned} \mathcal{M}_{\mu\nu} &= \frac{1}{2}\left\langle \widehat{q}_\mu \widehat{q}_\nu + \widehat{q}_\nu \widehat{q}_\mu \right\rangle - \left\langle q_\mu \right\rangle \left\langle q_\nu \right\rangle, \\ \sigma_a &= \langle \widehat{a}^2 \rangle - \langle \widehat{a} \rangle^2, \\ \sigma_{ab} &= \frac{1}{2}\langle \widehat{a}\widehat{b} + \widehat{b}\widehat{a} \rangle - \langle \widehat{a} \rangle \langle \widehat{b} \rangle. \end{aligned} \tag{10}$$

Actually, there is no need to perform tedious calculations to invert matrix Λ due to the symplectic identities

$$\Lambda\Sigma\tilde{\Lambda} \equiv \Sigma, \qquad \Lambda^{-1} \equiv \Sigma\tilde{\Lambda}\Sigma^{-1} = -\Sigma\tilde{\Lambda}\Sigma, \tag{11}$$

which are immediate consequences of Eq. (6) and the symmetric properties $\mathcal{B} = \tilde{\mathcal{B}}$, $\Sigma = -\tilde{\Sigma}$. Consequently,

$$\mathcal{L} = \left\| \begin{array}{cc} \tilde{\lambda}_4 & -\tilde{\lambda}_2 \\ -\tilde{\lambda}_3 & \tilde{\lambda}_1 \end{array} \right\|. \tag{12}$$

The identities (11) are equivalent to the following identities for the blocks of matrix Λ:

$$\begin{aligned} &\lambda_4(t)\widetilde{\lambda}_1(t) - \lambda_3(t)\widetilde{\lambda}_2(t) = I_N, \\ &\lambda_3(t)\widetilde{\lambda}_4(t) = \lambda_4(t)\widetilde{\lambda}_3(t), \\ &\widetilde{\lambda}_1(t)\lambda_3(t) = \widetilde{\lambda}_3(t)\lambda_1(t). \end{aligned} \tag{13}$$

3. Invariant Squeezing Coefficient

The instantaneous values of variances $\sigma_x(t)$ and $\sigma_p(t)$ cannot serve as true measures of squeezing in all the cases, since they periodically oscillate in the course of free evolution of each mode. For example, for an isolated mode with the frequency ω

$$\sigma_x(t) = \sigma_x(0)\cos^2(\omega t) + \sigma_p(0)\sin^2(\omega t) + \sigma_{xp}(0)\sin(2\omega t).$$

Therefore, even if $\sigma_x(0)$ is small, both values $\sigma_x(t)$ and $\sigma_p(t)$ can become large at some instant of time t. Or, on the contrary, it can happen that both "initial" variances $\sigma_x(0)$ and $\sigma_p(0)$ are large but, nonetheless, the state is, in fact, highly squeezed, because the contributions of $\sigma_x(0)$ and $\sigma_p(0)$ can be compensated at some moment by the large nonzero covariance $\sigma_{xp}(0)$. It is reasonable to introduce some invariant characteristics that do not depend on time in the course of free evolution. Looking for extrema values of $\sigma_x(t)$ as a function of time, one can easily find the following minimum σ_- and maximum σ_+ values of the variances σ_x or σ_p [38]:

$$\sigma_\pm = \mathcal{E} \pm \sqrt{\mathcal{E}^2 - \mathcal{D}} = \frac{\mathcal{D}}{\mathcal{E} \mp \sqrt{\mathcal{E}^2 - \mathcal{D}}}, \qquad \mathcal{E} = \frac{1}{2}\left(\sigma_x + \sigma_p\right), \qquad \mathcal{D} = \sigma_x\sigma_p - \sigma_{xp}^2.$$

Similar expressions were obtained in [39–41]. We define the invariant squeezing coefficient $\mathcal{S}$ as the ratio of σ_- to the vacuum dimensionless variance 1/2. In terms of the elements of the covariance matrix $\mathcal{M}$,

$$\mathcal{S}_k = 2\left[\mathcal{E}_k - \sqrt{\mathcal{E}_k^2 - \mathcal{D}_k}\right] = 2\mathcal{D}_k\left[\mathcal{E}_k + \sqrt{\mathcal{E}_k^2 - \mathcal{D}_k}\right]^{-1}, \tag{14}$$

where

$$\mathcal{E}_k = \frac{1}{2}\left[\mathcal{M}_{pp}^k + \mathcal{M}_{xx}^k\right] \tag{15}$$

is the energy of quantum fluctuations in the kth mode normalized by the quantum energy $\hbar\omega_k$, and

$$\mathcal{D}_k = \mathcal{M}_{pp}^{(k)}\mathcal{M}_{xx}^{(k)} - \left[\,\mathcal{M}_{xp}^{(k)}\right]^2 \tag{16}$$

is the invariant uncertainty product [42, 43] of the kth mode, which must satisfy the Schrödinger–Robertson uncertainty relation [37, 42, 44, 45] $\mathcal{D}_k \ge 1/4$. In particular, the states with large energy of fluctuations are always strongly squeezed, if the invariant uncertainty product is much less than this energy: $\mathcal{S} \approx \mathcal{D}/\mathcal{E} \ll 1$ if $\mathcal{E} \gg \mathcal{D}$ (this is always true for the Gaussian pure states, for which $\mathcal{D} \equiv 1/4$ [42]).

We suppose that there are no correlations between the modes in the initial states. Then the initial covariance matrix can be written in the form

$$\mathcal{M} = \frac{1}{2}\begin{bmatrix} \mathcal{P}_1 & 0 & \mathcal{R}_1 & 0 \\ 0 & \mathcal{P}_2 & 0 & \mathcal{R}_2 \\ \mathcal{R}_1 & 0 & \mathcal{X}_1 & 0 \\ 0 & \mathcal{R}_2 & 0 & \mathcal{X}_2 \end{bmatrix}, \tag{17}$$

with elements satisfying the Schrödinger–Robertson uncertainty relations for each mode:

$$\mathcal{P}_k\mathcal{X}_k - \mathcal{R}_k^2 \equiv 4\mathcal{D}_k^{(0)} \ge 1.$$

Sometimes, it can be useful to parametrize the initial matrix elements as follows:

$$\begin{aligned} \mathcal{P}_k &= 2\sqrt{\mathcal{D}_k^{(0)}}\left[\cosh 2\varrho_k + \sinh 2\varrho_k \cos(\vartheta_k)\right], \\ \mathcal{X}_k &= 2\sqrt{\mathcal{D}_k^{(0)}}\left[\cosh 2\varrho_k - \sinh 2\varrho_k \cos(\vartheta_k)\right] \end{aligned} \tag{18}$$

and

$$\begin{aligned} \mathcal{R}_k &= -2\sqrt{\mathcal{D}_k^{(0)}}\sinh 2\varrho_k \sin(\vartheta_k), \\ \mathcal{U}_k &\equiv \mathcal{P}_k - \mathcal{X}_k = 4\sqrt{\mathcal{D}_k^{(0)}}\sinh 2\varrho_k \cos(\vartheta_k). \end{aligned} \tag{19}$$

The initial squeezing coefficient and the energy of fluctuations do not depend on the phase ϑ:

$$\begin{aligned} S_k^{(0)} &= 2\sqrt{\mathcal{D}_k^{(0)}}\exp(-2|\varrho_k|), \\ \mathcal{E}_k^{(0)} &= \frac{1}{4}\left(\mathcal{P}_k + \mathcal{X}_k\right) = \sqrt{\mathcal{D}_k^{(0)}}\,\cosh 2\varrho_k. \end{aligned} \tag{20}$$

Another important phase-independent quantity is

$$\left[\mathcal{E}_k^{(0)}\right]^2 - \mathcal{D}_k^{(0)} \equiv \frac{1}{16}\left(\mathcal{U}_k^2 + 4\mathcal{R}_k^2\right) = \mathcal{D}_k^{(0)}\,\sinh^2(2\varrho_k). \tag{21}$$

Hereafter, we assume that the coefficients ϱ_k are nonnegative, making the phase parameters ϑ_k responsible for possible changes of the sign in Eqs. (18) and (19).

4. Entanglement Coefficients

Let us introduce the notation

$$\overline{ab} \equiv \left\langle \widehat{a}\widehat{b} \right\rangle - \langle \widehat{a} \rangle \left\langle \widehat{b} \right\rangle \tag{22}$$

for the unsymmetrized centered second-order moments of the operators $\widehat{a}$ and $\widehat{b}$. In the course of time, interacting modes become entangled and the cross-covariances $\overline{x_1x_2}$, $\overline{p_1p_2}$, $\overline{x_1p_2}$, and $\overline{x_2p_1}$ become different from zero. Each of these covariances alone is not a good measure of entanglement because of fast oscillations with the frequencies ω_1 and ω_2. However, one can verify, that in the absence of coupling in the Hamiltonian (1), the combinations

$$\begin{aligned} A &= \overline{p_1p_2} + \overline{x_1x_2}\,, \\ B &= \overline{x_1p_2} - \overline{p_1x_2}\,, \\ C &= \overline{x_1x_2} - \overline{p_1p_2}\,, \\ D &= \overline{x_1p_2} + \overline{p_1x_2} \end{aligned}$$

are transformed in the same way as the components of two-dimensional vectors under orthogonal transformations:

$$\begin{aligned} A(t) &= A(0)\cos(\omega_- t) + B(0)\sin(\omega_- t)\,, \\ B(t) &= B(0)\cos(\omega_- t) - A(0)\sin(\omega_- t)\,, \\ C(t) &= C(0)\cos(\omega_+ t) + D(0)\sin(\omega_+ t)\,, \\ D(t) &= D(0)\cos(\omega_+ t) - C(0)\sin(\omega_+ t)\,, \end{aligned}$$

where

$$\omega_\pm = \omega_2 \pm \omega_1.$$

Consequently, the quantities $A^2 + B^2$ and $C^2 + D^2$ do not depend on time (if the interaction is switched off). Therefore, the quantity

$$F = \frac{1}{2}\left[A^2 + B^2 + C^2 + D^2\right] = \left(\overline{x_1x_2}\right)^2 + \left(\overline{p_1p_2}\right)^2 + \left(\overline{x_1p_2}\right)^2 + \left(\overline{p_1x_2}\right)^2 \tag{23}$$

seems to be a good characteristics of entanglement because it is nonnegative, does not depend on time in the absence of interactions, and contains all the cross-covariances on an equal footing. It is desirable to normalize F somehow in order to obtain a dimensionless measure of entanglement, which would be limited, say, by the unit value. To find a convenient normalization factor, let us introduce the non-Hermitian creation and annihilation operators as

$$\hat{a}_k = \frac{\hat{x}_k + i\hat{p}_k}{\sqrt{2}}, \qquad k = 1, 2. \tag{24}$$

In terms of the new operators, we have

$$\begin{aligned} A &= \overline{a_1a_2^\dagger} + \overline{a_1^\dagger a_2}\,, \\ B &= i\left(\overline{a_1a_2^\dagger} - \overline{a_1^\dagger a_2}\right), \\ C &= \overline{a_1a_2} + \overline{a_1^\dagger a_2^\dagger}\,, \\ D &= i\left(\overline{a_1^\dagger a_2^\dagger} - \overline{a_1a_2}\right), \end{aligned}$$

so that

$$F = 2|\overline{a_1 a_2^\dagger}|^2 + 2|\overline{a_1 a_2}|^2.$$

The following inequalities (the special cases of a large family of the generalized uncertainty relations) exist [37] (note the order of operators under the averaging symbols on the right-hand side):

$$\overline{a_1 a_2^\dagger}\,\overline{a_1^\dagger a_2} \le \overline{a_1^\dagger a_1}\,\overline{a_2^\dagger a_2} \tag{25}$$

and

$$\overline{a_1 a_2}\,\overline{a_1^\dagger a_2^\dagger} \le \overline{a_1^\dagger a_1}\,\overline{a_2 a_2^\dagger}, \qquad \overline{a_1 a_2}\,\overline{a_1^\dagger a_2^\dagger} \le \overline{a_1 a_1^\dagger}\,\overline{a_2^\dagger a_2}. \tag{26}$$

The two inequalities in (26) can be combined into a single (although weaker) inequality, whose right-hand side is symmetric with respect to the change of indices $1 \leftrightarrow 2$ (as well as the left-hand side):

$$\overline{a_1 a_2}\,\overline{a_1^\dagger a_2^\dagger} \le \overline{a_1^\dagger a_1}\,\overline{a_2^\dagger a_2} + \frac{1}{2}\left(\overline{a_1^\dagger a_1} + \overline{a_2^\dagger a_2}\right). \tag{27}$$

Consequently,

$$|\overline{a_1 a_2^\dagger}|^2 + |\overline{a_1 a_2}|^2 \le 2\overline{a_1^\dagger a_1}\,\overline{a_2^\dagger a_2} + \frac{1}{2}\left(\overline{a_1^\dagger a_1} + \overline{a_2^\dagger a_2}\right) = 2\mathcal{E}_1\mathcal{E}_2 - \frac{1}{2}\left(\mathcal{E}_1 + \mathcal{E}_2\right) < 2\mathcal{E}_1\mathcal{E}_2, \tag{28}$$

where

$$\mathcal{E}_k = \overline{a_k^\dagger a_k} + \frac{1}{2}$$

is the same energy of fluctuations as defined in Eq. (15) in terms of the quadrature variances. Taking into account the last inequality in (28), it seems reasonable to define the dimensionless invariant entanglement coefficient as follows:

$$\mathcal{Y} = \left[\frac{(\overline{x_1 x_2})^2 + (\overline{p_1 p_2})^2 + (\overline{x_1 p_2})^2 + (\overline{p_1 x_2})^2}{4\mathcal{E}_1\mathcal{E}_2}\right]^{1/2} = \left[\frac{|\overline{a_1 a_2^\dagger}|^2 + |\overline{a_1 a_2}|^2}{2\left(\overline{a_1^\dagger a_1} + 1/2\right)\left(\overline{a_2^\dagger a_2} + 1/2\right)}\right]^{1/2}. \tag{29}$$

It satisfies the inequality $0 \le \mathcal{Y} < 1$. The upper limit 1 cannot be attained. Using the normalization factor in the denominator of (29) in the form $4\mathcal{E}_1\mathcal{E}_2 - \mathcal{E}_1 - \mathcal{E}_2$, we could obtain an exact equality $\mathcal{Y} = 1$ for some states. However, such a definition would result in serious problems for states whose energies of fluctuations are close to the minimum value $\mathcal{E}_{\min} = 1/2$ because, in this case, the denominator could turn to zero, resulting in an indefiniteness of the fraction.

Another motivation for the choice of the measure of entanglement in the form (29) is the possibility to express it in an elegant matrix form. For this purpose, it is convenient to rearrange the components of the vector $\mathbf{q}$ defined in Eq. (2) by introducing the vector

$$\mathbf{z} = (p_1, x_1, p_2, x_2).$$

The rearranged covariance matrix

$$\mathcal{Z} \equiv \frac{1}{2}\,\|\overline{z_\mu z_\nu} + \overline{z_\nu z_\mu}\|$$

has the following block structure:

$$\mathcal{Z} = \left\|\begin{array}{cc} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ \mathcal{Z}_{21} & \mathcal{Z}_{22} \end{array}\right\|, \qquad \mathcal{Z}_{11} = \tilde{\mathcal{Z}}_{11}, \qquad \mathcal{Z}_{22} = \tilde{\mathcal{Z}}_{22}, \qquad \mathcal{Z}_{12} = \tilde{\mathcal{Z}}_{21}.$$

Obviously, the information concerning entanglement is contained in the "cross-covariance" blocks $\mathcal{Z}_{12}$ and $\mathcal{Z}_{21}$. It was shown recently that the determinant ($\det \mathcal{Z}_{12}$) of these blocks plays an important role for the problem of separability [46–50]. One can easily check that

$$\det \mathcal{Z}_{12} = \frac{1}{4}\left[A^2+B^2-C^2-D^2\right] = |\overline{a_1 a_2^\dagger}|^2 - |\overline{a_1 a_2}|^2.$$

On the other hand, one can verify that the quantity F defined in Eq. (23) can be written as

$$F = \mathrm{Tr}\left(\mathcal{Z}_{12}\mathcal{Z}_{21}\right).$$

Taking into account that

$$\mathcal{E}_k = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{Z}_{kk},$$

one can write the parameter $\mathcal{Y}$ in a compact matrix form

$$\mathcal{Y} = \left[\frac{\mathrm{Tr}\left(\mathcal{Z}_{12}\mathcal{Z}_{21}\right)}{\mathrm{Tr}\,\mathcal{Z}_{11}\,\mathrm{Tr}\,\mathcal{Z}_{22}}\right]^{1/2}. \tag{30}$$

Using the relationship

$$2\sqrt{\mathcal{E}_1\mathcal{E}_2} \le \mathcal{E}_1 + \mathcal{E}_2 = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{Z},$$

we can introduce another entanglement parameter

$$\tilde{\mathcal{Y}} = \frac{2\sqrt{\mathrm{Tr}\left(\mathcal{Z}_{12}\mathcal{Z}_{21}\right)}}{\mathrm{Tr}\mathcal{Z}} = \frac{\sqrt{2\left(|\overline{a_1 a_2^\dagger}|^2 + |\overline{a_1 a_2}|^2\right)}}{\overline{a_1^\dagger a_1} + \overline{a_2^\dagger a_2} + 1}, \tag{31}$$

which obeys the inequality $\tilde{\mathcal{Y}} \le \mathcal{Y}$. However, in this paper we shall consider the parameter $\mathcal{Y}$ only.

It should be noted that there exist many different definitions of the measure of entanglement. As a rule, they are based on different kinds of entropies, mainly on the von Neumann entropy and its modifications [51–56]. For example, if the total system, consisting of parts 1 and 2, is decribed by means of the statistical operator $\hat{\rho}_0$, then the measure of entanglement is frequently expressed in terms of the total and "partial" entropies as the "index of correlation" [51]

$$\mathcal{I}_c = \mathcal{I}_1 + \mathcal{I}_2 - \mathcal{I}_0, \qquad \mathcal{I}_k = -\mathrm{Tr}_k\,\hat{\rho}_k\,\ln\hat{\rho}_k, \tag{32}$$

where the reduced statistical operator is defined as, e.g.,

$$\hat{\rho}_1 = \mathrm{Tr}_2\,\hat{\rho}_0.$$

More simple, from the viewpoint of calculations, are measures based on the "quantum purity" $\mathrm{Tr}\,\hat{\rho}^2$, which were considered, e.g., in [57–60] (although only in the case of finite-dimensional Hilbert spaces). For the "continuous variable systems" and in the case where the total system is in a pure quantum state, it seems reasonable to consider the "linear entropy of entanglement" defined as

$$\mathcal{K} = 1 - \mathrm{Tr}_1\,\hat{\rho}_1^2 = 1 - \mathrm{Tr}_2\,\hat{\rho}_2^2. \tag{33}$$

For the Gaussian states, it depends on the invariant uncertainty product (16) only

$$\mathcal{K}_G \equiv K = 1 - (4\mathcal{D}_1)^{-1/2} = 1 - (4\mathcal{D}_2)^{-1/2}. \tag{34}$$

For non-Gaussian states, expressions (33) and (34) are not equivalent. For example, taking the Fock states with sufficiently large numbers, one can obtain values as large as desired for the product $\mathcal{D}$, although all such states have the same purity

$$\mathrm{Tr}\,\hat{\rho}^2 = 1.$$

Moreover, there exists an interesting generalization of the uncertainty relation in the form [37, 61]

$$\sqrt{\mathcal{D}} \geq \frac{1}{2}\, f\left(\mathrm{Tr}\,\hat{\rho}^2\right),$$

where the function $f(\mu)$ equals 1 for $\mu = 1$ and goes to ∞ if $\mu \to 0$, being less than μ^{-1}. Asymptotically, as $\mu \to 0$, $f(\mu) \approx 8/(9\mu)$.

The von Neumann entropy of the Gaussian states of each mode also depends only on the invariant uncertainty product (16) or on the linear entropy K [62, 63]:

$$I_1 = \left(\sqrt{\mathcal{D}_1} + \frac{1}{2}\right) \ln \left(\sqrt{\mathcal{D}_1} + \frac{1}{2}\right) - \left(\sqrt{\mathcal{D}_1} - \frac{1}{2}\right) \ln \left(\sqrt{\mathcal{D}_1} - \frac{1}{2}\right), \tag{35}$$

$$I_1 = \ln \frac{2-K}{2(1-K)} + \frac{K}{2(1-K)} \ln \frac{2-K}{K}. \tag{36}$$

If $K \to 0$ (almost pure states), then

$$I \approx \frac{K}{2} \ln \left(\frac{2}{K}\right),$$

whereas for highly mixed states with $1 - K \ll 1$, we have

$$\exp\left(-I\right) \approx \frac{2}{e}\left(1 - K\right).$$

In contrast to Eqs. (34) and (35), formulas (29) or (30) hold for arbitrary states. The possibilities of using the Hilbert–Schmidt distance between the given state and different "disentangled" states were discussed, e.g., in [55, 64], and a simple formula was given recently in [65] (an analogous approach was developed in [66] for the quantify called by the authors the "degree of nonclassicality" of quantum states).

5. Evolution of Squeezing and Entanglement Coefficients

To find the time dependences of squeezing and entanglement coefficients for two coupled oscillators, we have to solve Eq. (6), which is equivalent to the following system of linear differential equations and initial conditions for the matrices $\lambda_j(t)$:

$$\dot{\lambda}_1 = \lambda_1\, b_3 - \lambda_2\, b_1, \qquad \lambda_1(0) = I_N, \tag{37}$$

$$\dot{\lambda}_2 = \lambda_1\, b_4 - \lambda_2\, b_2, \qquad \lambda_2(0) = 0, \tag{38}$$

$$\dot{\lambda}_3 = \lambda_3\, b_3 - \lambda_4\, b_1, \qquad \lambda_3(0) = 0, \tag{39}$$

$$\dot{\lambda}_4 = \lambda_3\, b_4 - \lambda_4\, b_2, \qquad \lambda_4(0) = I_N. \tag{40}$$

For nonsingular matrix b_1, one can exclude matrices λ_2 and λ_4, arriving at the identical second-order equations for the matrices λ_3 and λ_1:

$$\frac{d^2\lambda_3}{dt^2} - \frac{d\lambda_3}{dt}G_3 + \lambda_3 G_4 = 0, \tag{41}$$

where

$$\begin{aligned} G_3 &= b_3 - b_1^{-1}b_2b_1 + b_1^{-1}\frac{db_1}{dt}\,, \\ G_4 &= b_4b_1 - b_3b_1^{-1}b_2b_1 + b_3b_1^{-1}\frac{db_1}{dt} - \frac{db_3}{dt}\,. \end{aligned}$$

The initial conditions are

$$\lambda_1(0) = I_N, \qquad \dot\lambda_1(0) = b_3, \qquad \lambda_3(0) = 0, \qquad \dot\lambda_3(0) = -b_1.$$

In the case of Hamiltonian (1), the matrices b_j read

$$b_1 = \left\| \begin{matrix} \omega_1 & \varpi\gamma_1(t) \\ \varpi\gamma_1(t) & \omega_2 \end{matrix} \right\|, \qquad b_4 = \left\| \begin{matrix} \omega_1 & \varpi\gamma_4(t) \\ \varpi\gamma_4(t) & \omega_2 \end{matrix} \right\|, \qquad b_2 = \tilde b_3 = \left\| \begin{matrix} 0 & \varpi\gamma_2(t) \\ \varpi\gamma_3(t) & 0 \end{matrix} \right\|.$$

One can consider each row of matrix λ_3 (or λ_1) as a two-dimensional vector

$$\mathbf{v} = (\lambda_{l1}, \lambda_{l2}) \equiv (v_1, v_2)\,, \qquad l = 1, 2.$$

Therefore, the matrix equation (41) is equivalent to the vector equation

$$\ddot{\mathbf{v}} + \mathcal{M}\mathbf{v} = \mathcal{N}(t)\dot{\mathbf{v}} + \mathcal{Q}(t)\mathbf{v} \tag{42}$$

with the diagonal matrix

$$\mathcal{M} = \mathrm{diag}\left(\omega_1^2, \omega_2^2\right)$$

on the left-hand side and antidiagonal matrices

$$\mathcal{N} = \left\| \begin{matrix} 0 & h_1 \\ h_3 & 0 \end{matrix} \right\|, \qquad \mathcal{Q} = \left\| \begin{matrix} 0 & -h_2 \\ -h_4 & 0 \end{matrix} \right\|,$$

where

$$h_1 = \varpi\omega_2^{-1}\Big[(\omega_2\gamma_2 - \omega_1\gamma_3) + \dot\gamma_1\Big], \qquad h_2 = \varpi\Big[(\omega_1\gamma_4 + \omega_2\gamma_1) - \dot\gamma_2\Big], \tag{43}$$

$$h_3 = \varpi\omega_1^{-1}\Big[(\omega_1\gamma_3 - \omega_2\gamma_2) + \dot\gamma_1\Big], \qquad h_4 = \varpi\Big[(\omega_2\gamma_4 + \omega_1\gamma_1) - \dot\gamma_3\Big], \tag{44}$$

on the right-hand side. Here, we have neglected terms proportional to squares and products of the coupling constants.

In this paper, we consider the case of weak resonance coupling, i.e., we assume that the coupling coefficients are small, $|\gamma_k| \ll 1$, and the frequency η equals either $\omega_1 + \omega_2$ (the parametric amplifier) or $\omega_1 - \omega_2$ (the parametric converter). In this case, we can use, e.g., the Bogoliubov–Mitropolsky scheme [67] to find approximate analytical solutions to Eq. (42). Shortly, this scheme is as follows (see Appendix A for more details). We introduce slowly varying in time amplitudes z_j and z_{-j} according to the relationships:

$$v_j = z_j e^{i\omega_j t} + z_{-j} e^{-i\omega_j t}, \tag{45}$$

$$\dot{v}_j = i\omega_j \left(z_j e^{i\omega_j t} - z_j e^{-i\omega_j t} \right). \tag{46}$$

Putting these expressions into Eq. (42) and using the method of averaging (i.e., multiplying the equations by the factors $\exp(\pm i\omega_j t)$ and averaging the resulting expressions over the period of fast oscillations with frequency ω_j), one can obtain for the functions $z_j(t)$ and $z_{-j}(t)$ a set of first-order linear differential equations with constant coefficients, all of which have an order of small coupling parameters γ_k. The explicit forms of these equations, are given in Appendix A, whereas the consequences of their solutions, namely, the elements of the time-dependent covariance matrices are given in Appendix B.

In the following sections, we confine ourselves to the analysis of the squeezing and entanglement coefficients, considering separately the cases of parametric amplifier and parametric converter. For the sake of simplicity, we give the explicit expressions in the simplest case of equal phases ϕ_k where all of them can be assumed to be equal to zero, showing at the end of the section how the formulas should be modified in the generic case.

5.1. Parametric Amplifier

If $\eta = \omega_1 + \omega_2$, then the energies of fluctuations and the invariant uncertainty products increase with time exponentially. Using the $(\varrho - \vartheta)$-parametrization of the initial covariance matrix, we obtain

$$\mathcal{E}_1(\tilde{\tau}) = \mathcal{E}_1^{(0)} \cosh^2 \tilde{\tau} + \mathcal{E}_2^{(0)} \sinh^2 \tilde{\tau}, \tag{47}$$

$$\begin{aligned} \mathcal{D}_1(\tilde{\tau}) &= \mathcal{D}_1^{(0)} \cosh^4 \tilde{\tau} + \mathcal{D}_2^{(0)} \sinh^4 \tilde{\tau} \\ &\quad + 2 \sinh^2 \tilde{\tau} \cosh^2 \tilde{\tau} \sqrt{\mathcal{D}_1^{(0)} \mathcal{D}_2^{(0)}} \Big[\cosh 2\varrho_1 \cosh 2\varrho_2 - \widetilde{\sigma} \sinh 2\varrho_1 \sinh 2\varrho_2 \Big], \end{aligned} \tag{48}$$

and

$$\begin{aligned} \mathcal{E}_1^2 - \mathcal{D}_1 &= \mathcal{D}_1^{(0)} \sinh^2(2\varrho_1) \cosh^4 \tilde{\tau} + \mathcal{D}_2^{(0)} \sinh^2(2\varrho_2) \sinh^4 \tilde{\tau} \\ &\quad + 2\widetilde{\sigma} \sqrt{\mathcal{D}_1^{(0)} \mathcal{D}_2^{(0)}} \sinh(2\varrho_1) \sinh(2\varrho_2) \sinh^2 \tilde{\tau} \cosh^2 \tilde{\tau}, \end{aligned} \tag{49}$$

where

$$\widetilde{\sigma} = \cos(2\widetilde{\varphi} - \vartheta_1 - \vartheta_2), \tag{50}$$

$$\sin \widetilde{\varphi} = \frac{\widetilde{\mu}_0}{\sqrt{\widetilde{\mu}_0^2 + \widetilde{\nu}_0^2}}, \qquad \cos \widetilde{\varphi} = \frac{\widetilde{\nu}_0}{\sqrt{\widetilde{\mu}_0^2 + \widetilde{\nu}_0^2}}, \tag{51}$$

and

$$\widetilde{\mu}_0 = \gamma_1 - \gamma_4, \qquad \widetilde{\nu}_0 = \gamma_2 + \gamma_3, \qquad \widetilde{\kappa} = \frac{1}{4}\sqrt{\widetilde{\mu}_0^2 + \widetilde{\nu}_0^2}, \qquad \tilde{\tau} = \varpi \widetilde{\kappa} t. \tag{52}$$

For the second mode, one should change the indices $1 \leftrightarrow 2$.

We confine our analysis of Eqs. (47)–(49) to the case of initial pure states with

$$\mathcal{D}_1^{(0)} = \mathcal{D}_2^{(0)} = \frac{1}{4}\,.$$

Two special cases where $\widetilde{\sigma} = \pm 1$ are the most interesting not only because they correspond to the extremum values of all parameters but also because in these cases the expression on the right-hand side of (49) becomes the full square, thus enabling us to find simple explicit final formulas.

If $\widetilde{\sigma} = 1$ then, using the first equality in Eq. (14), we obtain the following simple expression for the squeezing coefficient:

$$S_1 = \cosh^2\tilde{\tau}\, e^{-2\varrho_1} + \sinh^2\tilde{\tau}\, e^{-2\varrho_2}. \tag{53}$$

If $\widetilde{\sigma} = -1$, then one should remember that $\sqrt{a^2} = |a|$, which is important for $a < 0$.

If $\varrho_1 < \varrho_2$, then we have two different analytical expressions for S_1 in the intervals $0 \le \tilde{\tau} \le \tilde{\tau}_*$ and $\tilde{\tau}_* \le \tilde{\tau} < \infty$, where $\tilde{\tau}_*$ is the instant when the right-hand side of (49) turns to zero:

$$S_1 = \begin{cases} \cosh^2\tilde{\tau}\, e^{-2\varrho_1} + \sinh^2\tilde{\tau}\, e^{2\varrho_2}, & 0 \le \tilde{\tau} \le \tilde{\tau}_* \\ \cosh^2\tilde{\tau}\, e^{2\varrho_1} + \sinh^2\tilde{\tau}\, e^{-2\varrho_2}, & \tilde{\tau}_* \le \tilde{\tau} < \infty \end{cases}, \tag{54}$$

$$\tanh^2\tilde{\tau}_* = \frac{\sinh(2\varrho_1)}{\sinh(2\varrho_2)}\,. \tag{55}$$

We see that S_1 monotonically increases with time for any ϱ_1 and ϱ_2. Note that the derivative $dS_1/d\tilde{\tau}$ has a discontinuity at $\tilde{\tau} = \tilde{\tau}_*$. At this point, we have

$$S_1(\tilde{\tau}_*) = \frac{\sinh(\varrho_1 + \varrho_2)}{\sinh(\varrho_2 - \varrho_1)}, \qquad \left.\frac{dS_1}{d\tilde{\tau}}\right|_{\tilde{\tau}*-0} - \left.\frac{dS_1}{d\tilde{\tau}}\right|_{\tilde{\tau}*+0} = 4\sqrt{\sinh(2\varrho_1)\sinh(2\varrho_2)}.$$

In a special case of the initial coherent state of the first mode ($\varrho_1 = 0$), we have

$$\mathcal{S}_1(\tilde{\tau}) = 1 + \left(1 + e^{-2\varrho_2}\right)\sinh^2\tilde{\tau} \tag{56}$$

independently of the parameters ϑ_2 and $\widetilde{\varphi}$, i.e., on the covariance R_2 and on the type of the bilinear coupling. Consequently, no bilinear coupling can squeeze the coherent state or improve the squeezing in the case of parametric amplification under the exact resonance condition

$$\eta = \omega_1 + \omega_2.$$

This is a generalization of the result obtained in [23] for the special case of coordinate coupling x_1x_2.

The entanglement coefficient depends on the "slow time" $\tilde{\tau}$ as follows (for pure initial quantum states)

$$\mathcal{Y} = \frac{\sinh(2\tilde{\tau})}{2}\sqrt{\frac{2\cosh^2(\varrho_1 - \varrho_2)\cosh 2(\varrho_1 + \varrho_2) + (\widetilde{\sigma} - 1)\sinh 2\varrho_2 \sinh 2\varrho_1}{\cosh 2\varrho_1 \cosh 2\varrho_2 + \sinh^2(2\tilde{\tau})\cosh^2(\varrho_1 + \varrho_2)\cosh^2(\varrho_1 - \varrho_2)}}\,. \tag{57}$$

Asymptotically, as $\tilde{\tau} \to \infty$, the entanglement coefficient tends to a constant value (provided the parameter $\widetilde{\kappa}$ is different from zero), whose dependence on the coupling coefficients of the Hamiltonian is "hidden" in the angle $\widetilde{\varphi}$ or in the parameter $\widetilde{\sigma}$:

$$\mathcal{Y}_\infty = \frac{\sqrt{2\cosh^2(\varrho_1 - \varrho_2)\cosh 2(\varrho_1 + \varrho_2) + (\widetilde{\sigma} - 1)\sinh 2\varrho_2 \sinh 2\varrho_1}}{2\cosh(\varrho_1 + \varrho_2)\cosh(\varrho_1 - \varrho_2)}\,. \tag{58}$$

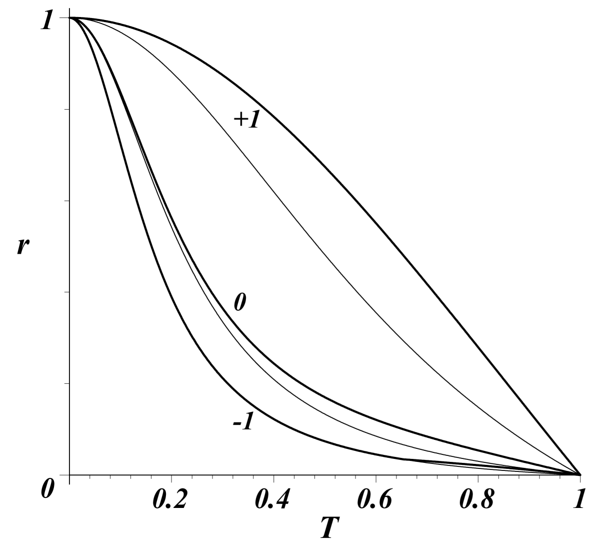

Fig. 1. The inverse squeezing coefficient $r(T) = S(0)/S(T)$ versus the "compact slow time" $T = \tanh\tilde{\tau}$ in the case of a parametric amplifier. For each label (which gives the value of the parameter $\widetilde{\sigma}$), the upper curves correspond to r_1 and the lower curves correspond to r_2 (for $\widetilde{\sigma} = -1$ the two curves are practically indistinguishable for $T < 0.65$). The initial squeezing parameters are $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$.

Since $-1 \le \widetilde{\sigma} \le 1$, depending on the concrete values of parameters $\widetilde{\varphi}$ and ϑ_k, the asymptotical entanglement coefficient is confined in the interval

$$\sqrt{\frac{1}{2}\left[1+\tanh^2\left(\varrho_1-\varrho_2\right)\right]} \le \mathcal{Y}_\infty \le \sqrt{\frac{1}{2}\left[1+\tanh^2\left(\varrho_1+\varrho_2\right)\right]}. \tag{59}$$

Consequently, $\mathcal{Y}_\infty$ always exceeds the value $\sqrt{2}/2 \approx 0.7$ (the minimum is achieved for $\varrho_1 = \varrho_2$ and $\widetilde{\sigma} = -1$).

If $\varrho_k \gg 1$ for $k = 1$ or $k = 2$, then the maximum asymptotic value of the entanglement coefficient (this can be obtained if $\widetilde{\sigma} = 1$) is very close to unity.

If one of the modes was initially in the coherent state ($\varrho_1 = 0$ or $\varrho_2 = 0$), then neither the squeezing coefficients nor the entanglement coefficient depend on the phase $\widetilde{\varphi}$.

The results obtained in this section are illustrated in Figs. 1–3.

It is convenient to use, instead of the slow time $\tilde{\tau}$ and the squeezing coefficient S (which can vary from zero to ∞), the "compact time" $T = \tanh\tilde{\tau}$ and the normalized inverse squeezing coefficient $r(T) = S(0)/S(T)$, having in mind that the squeezing coefficient can grow unlimitedly in the case of parametric amplifier.

In Fig. 1, we show typical dependences $r(T)$ for three values of parameter $\widetilde{\sigma} = 1, 0, -1$, whereas in Fig. 2 we give the entanglement coefficient $\mathcal{Y}(T)$ for the same values of parameters. In the case $\widetilde{\sigma} = -1$, the curves for $r_1(T)$ and $r_2(T)$ practically coincide until $T = T_* = \sqrt{\sinh(2\varrho_1)/\sinh(2\varrho_2)} \approx 0.65$ (at this point $r_1(T_*) \approx r_2(T_*) \approx 0.03$) where the derivative of the function $r_1(T)$ has a jump, slowing down the decay of $r_1(T)$, whereas the derivative of $r_2(T)$ has no discontinuity.

In Fig. 3, we compare (for the Gaussian states) the linear entropy of entanglement $K(T)$ (34) and

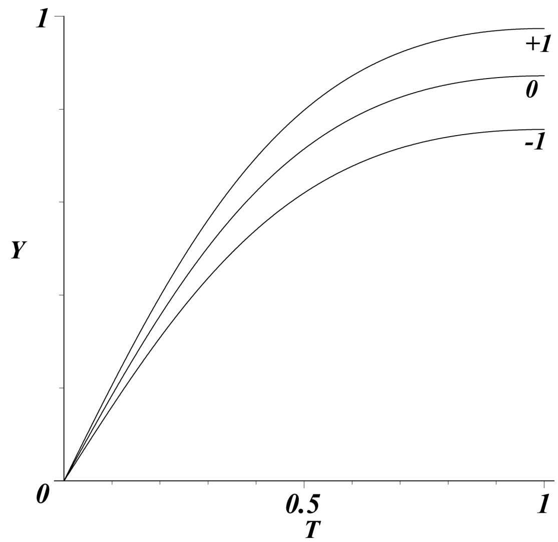

Fig. 2. The covariance entanglement coefficient $\mathcal{Y}$ versus the "compact slow time" $T = \tanh\tilde{\tau}$ in the case of a parametric amplifier. The numbers below the curves give the corresponding values of the parameter $\widetilde{\sigma}$. The initial squeezing parameters are $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$.

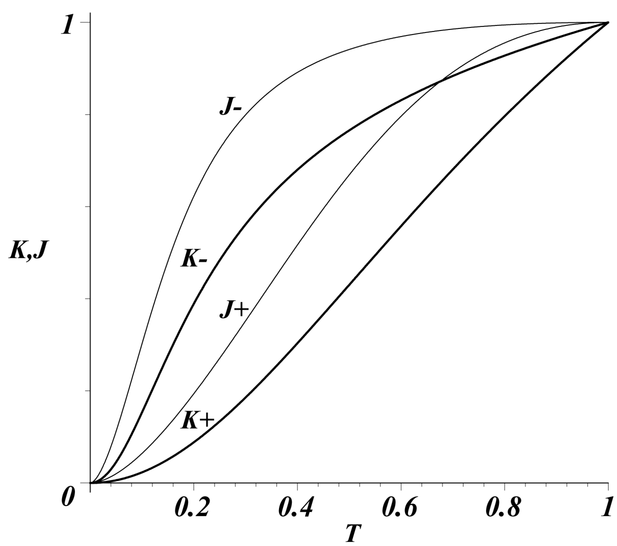

Fig. 3. The linear entropy $K_1 = K_2$ (34) and the "compact entropy" $J_1 = J_2$ (60) of initially pure Gaussian states versus the "compact slow time" $T = \tanh\tilde{\tau}$ in the case of a parametric amplifier. The curves labeled "$J-$" and "$K-$" correspond to $\widetilde{\sigma} = -1$, whereas the curves labeled "$J+$" and "$K+$" correspond to $\widetilde{\sigma} = 1$. The initial squeezing parameters are $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$.

the "compact entropy"

$$J(T) = \tanh I(T) = \frac{\left(\sqrt{\mathcal{D}} + 1/2\right)^{2\sqrt{\mathcal{D}}+1} - \left(\sqrt{\mathcal{D}} - 1/2\right)^{2\sqrt{\mathcal{D}}-1}}{\left(\sqrt{\mathcal{D}} + 1/2\right)^{2\sqrt{\mathcal{D}}+1} + \left(\sqrt{\mathcal{D}} - 1/2\right)^{2\sqrt{\mathcal{D}}-1}} \tag{60}$$

for the extremal values $\widetilde{\sigma} = \pm 1$, for which a simple expression

$$4\mathcal{D}_1 = 1 + \sinh^2(2\tilde{\tau}) \cosh^2(\varrho_1 - \widetilde{\sigma}\varrho_2)$$

can be obtained.

In this case,

$$1 - K \approx [\sinh(2\tilde{\tau}) \cosh(\varrho_1 - \widetilde{\sigma}\varrho_2)]^{-1}, \qquad I \sim 2\tilde{\tau}, \qquad \tilde{\tau} \gg 1,$$
$$K \approx 2\tilde{\tau}^2 \cosh^2(\varrho_1 - \widetilde{\sigma}\varrho_2), \qquad I \approx -\cosh^2(\varrho_1 - \widetilde{\sigma}\varrho_2)\tilde{\tau}^2 \ln\left(\tilde{\tau}^2\right), \qquad \tilde{\tau} \ll 1,$$

whereas $\mathcal{Y} \sim \tilde{\tau}$ for $\tilde{\tau} \ll 1$. One should pay attention to the inverse order of curves for $\mathcal{Y}(T)$ and $K(T)$ [or $J(T)$] with respect to the parameter $\widetilde{\sigma}$.

5.2. Parametric Converter

If $\eta = \omega_1 - \omega_2$ then, roughly speaking, the hyperbolic functions of time should be replaced by trigonometric ones. However, one should pay attention to some difference in the sign of the coefficients, because now the "long time" τ and the angle φ (without tildes) are defined by means of the relationships:

$$\sin\varphi = \frac{\mu_0}{\sqrt{\mu_0^2 + \nu_0^2}}, \qquad \cos\varphi = \frac{\nu_0}{\sqrt{\mu_0^2 + \nu_0^2}}, \qquad \sigma = \cos(2\varphi + \vartheta_1 - \vartheta_2), \tag{61}$$

$$\mu_0 = \gamma_1 + \gamma_4, \qquad \nu_0 = \gamma_2 - \gamma_3, \qquad \kappa = \frac{1}{4}\sqrt{\mu_0^2 + \nu_0^2}, \qquad \tau = \varpi\kappa t. \tag{62}$$

Formula (62) for κ holds provided the difference between the frequencies is not too small, namely, $|\eta| \gg \kappa$.

If $\eta = 0$ then all formulas below remain valid, but one should replace the factor 1/4 in the definition of κ by the factor 1/2. The explanation is given in Appendix A.

The "transition regime" (when $0 < |\eta| < \kappa$) requires a special study and is not considered here.

The energies of fluctuations and the invariant uncertainty products now oscillate with time:

$$\mathcal{E}_1 = \mathcal{E}_1^{(0)} \cos^2\tau + \mathcal{E}_2^{(0)} \sin^2\tau, \tag{63}$$

$$\begin{aligned} \mathcal{D}_1(\tau) \;=\;& \mathcal{D}_1^{(0)} \cos^4\tau + \mathcal{D}_2^{(0)} \sin^4\tau \\ &+ 2\sin^2\tau\cos^2\tau \sqrt{\mathcal{D}_1^{(0)}\mathcal{D}_2^{(0)}} \Big[\cosh 2\varrho_1 \cosh 2\varrho_2 - \sigma \sinh 2\varrho_1 \sinh 2\varrho_2\Big], \end{aligned} \tag{64}$$

and

$$\begin{aligned} \mathcal{E}_1^2 - \mathcal{D}_1 \;=\;& \mathcal{D}_1^{(0)} \sinh^2(2\varrho_1)\cos^4\tau + \mathcal{D}_2^{(0)} \sinh^2(2\varrho_2)\sin^4\tau \\ &+ 2\sigma\sqrt{\mathcal{D}_1^{(0)}\mathcal{D}_2^{(0)}} \sinh(2\varrho_1)\sinh(2\varrho_2)\sin^2\tau\cos^2\tau. \end{aligned} \tag{65}$$

If $\sigma = 1$, then a monotonous transformation of the squeezing coefficient from the initial value $\exp(-2\varrho_1)$ to the final value $\exp(-2\varrho_2)$ is observed in the interval $0 \le \tau \le \pi/2$

$$S_1(\tau) = \cos^2\tau\, e^{-2\varrho_1} + \sin^2\tau\, e^{-2\varrho_2}. \tag{66}$$

However, the picture is qualitatively different for $\sigma = -1$ because the dependence is no longer monotonous — first we have an increase in the interval $0 \le \tau \le \tau_*$ (for any values of ϱ_1 and ϱ_2) and then a monotonous decay to the value $\exp(-2\varrho_2)$:

$$S_1 = \begin{cases} \cos^2\tau\, e^{-2\varrho_1} + \sin^2\tau\, e^{2\varrho_2}, & 0 \le \tau \le \tau_* \\ \cos^2\tau\, e^{2\varrho_1} + \sin^2\tau\, e^{-2\varrho_2}, & \tau_* \le \tau \le \pi/2 \end{cases}, \tag{67}$$

$$\tan^2\tau_* = \frac{\sinh(2\varrho_1)}{\sinh(2\varrho_2)}. \tag{68}$$

A sharp maximum with a discontinuity of the derivative $dS_1/d\tau$ is observed at $\tau = \tau_*$:

$$S_1^{(\max)} = \frac{\cosh(\varrho_1+\varrho_2)}{\cosh(\varrho_1-\varrho_2)}, \qquad \left.\frac{dS_1}{d\tau}\right|_{\tau*-0} = \sin(2\tau_*)\left(e^{2\varrho_2} - e^{-2\varrho_1}\right), \qquad \left.\frac{dS_1}{d\tau}\right|_{\tau*+0} = \sin(2\tau_*)\left(e^{-2\varrho_2} - e^{2\varrho_1}\right).$$

However, the discontinuity of the derivative is a peculiar feature of the case $\sigma = -1$ only; if $\sigma > -1$, then a smooth parabolic behavior in the neighborhood of the maximum is observed.

Since $S_1^{(\max)} \ge 1$, the state at the moment τ_* is always unsqueezed. This is explained by the fast growth of the invariant uncertainty product, equivalent (for the Gaussian states) to the fast loss of quantum purity in each mode:

$$4\mathcal{D}_1 = 1 + \sin^2(2\tau)\sinh^2(\varrho_1 - \sigma\varrho_2), \qquad \sigma = \pm 1.$$

Only the states with the same initial degree of squeezing can remain pure in the process of evolution, provided the interaction Hamiltonian is chosen in such a way that $\sigma = 1$.

In a special case of the initial coherent state of the first mode ($\varrho_1 = 0$), we have

$$\mathcal{S}_1(\tau) = 1 - \left(1 - e^{-2\varrho_2}\right)\sin^2\tau \tag{69}$$

independently of the parameters ϑ_2 and φ.

Imposing certain additional conditions on the ratio of frequencies and the coupling coefficients, it is possible to obtain not only exchange of energies, squeezing coefficients, etc., but also an exchange of whole quantum states. This problem has been studied in detail in [35].

For $\varrho_1 \ne 0$, the initial evolution of the squeezing coefficient is given by the expansion

$$S_1(\tau) = S_1(0) + \tau^2\left[\cosh(2\varrho_2) - \sigma\sinh(2\varrho_2) - e^{-2\varrho_1}\right] + \mathcal{O}(\tau^4), \qquad -1 \le \sigma \le 1,$$

which holds provided $\tau^2 \ll \sinh(2\varrho_1)/\sinh(2\varrho_2)$.

The entanglement coefficient $\mathcal{Y}$ is now an oscillating function of the slow time τ with period π (the formulas below are given for pure initial quantum states):

$$\mathcal{Y} = \frac{|\sin(2\tau)|}{2}\sqrt{\frac{2\sinh^2(\varrho_1-\varrho_2)\cosh 2(\varrho_1+\varrho_2) + (1-\sigma)\sinh 2\varrho_2 \sinh 2\varrho_1}{\cosh 2\varrho_1 \cosh 2\varrho_2 + \sin^2(2\tau)\sinh^2(\varrho_1+\varrho_2)\sinh^2(\varrho_1-\varrho_2)}}. \tag{70}$$

Function (70) reaches the maximum for $|\sin(2\tau)| = 1$:

$$\mathcal{Y}_{\max} = \frac{\sqrt{2\sinh^2(\varrho_1-\varrho_2)\cosh 2(\varrho_1+\varrho_2) + (1-\sigma)\sinh 2\varrho_2\sinh 2\varrho_1}}{2\cosh(\varrho_1+\varrho_2)\cosh(\varrho_1-\varrho_2)}. \tag{71}$$

Depending on the concrete value of the parameter σ, the maximum value can vary in the interval

$$|\tanh(\varrho_1-\varrho_2)|\sqrt{\frac{1}{2}\left[1+\tanh^2(\varrho_1+\varrho_2)\right]}\le\mathcal{Y}_{\max}\le\tanh(\varrho_1+\varrho_2)\sqrt{\frac{1}{2}\left[1+\tanh^2(\varrho_1-\varrho_2)\right]}. \qquad (72)$$

If the initial degrees of squeezing of two modes coincide, $\varrho_1=\varrho_2=\varrho$, then

$$0\le\mathcal{Y}_{\max}\le\frac{\tanh(2\varrho)}{\sqrt{2}}.$$

Consequently, in such a case, the entanglement coefficient can never exceed the maximum value $1/\sqrt{2}$ for any initial state and any set of coupling coefficients. Moreover, it is possible to adjust the coupling coefficients in such a way that $\mathcal{Y}(\tau)\le\mathcal{Y}_{\max}\equiv 0$. Generally speaking, this does not imply that the modes will be disentangled for an arbitrary initial state, because the measure $\mathcal{Y}$ is not absolutely universal, in the sense that one can construct obviously entangled states, for which $\mathcal{Y}=0$ [this is quite clear from the second equality in the definition (29)]. However, if we confine ourselves to the Gaussian states (which remain Gaussian in the process of evolution governed by the quadratic Hamiltonians [37] and whose statistical properties are completely determined by the covariance matrix and the mean values of the quadratures [37, 68]), in particular, to the initial squeezed states, then the equality $\mathcal{Y}=0$ is the necessary and sufficient condition of disentanglement.

A particular situation occurs when we start from the initial coherent states for which $\varrho_1=\varrho_2=0$. Then $\mathcal{Y}(\tau)\equiv 0$ for any choice of coupling constants. Consequently, the parametric converter preserves coherent states. In particular, in the case of time-independent resonance coupling, $\eta=\omega_1-\omega_2=0$, any initial coherent state remains coherent in the process of evolution for arbitrary (small) coupling coefficients (provided the second-order corrections with respect to these small coefficients can be neglected). Also, as in the case of parametric amplifier, squeezing and entanglement coefficients do not depend on the phase φ if one of the modes was initially in the coherent state ($\varrho_1=0$ or $\varrho_2=0$).

The dependences $S(\tau)$, $\mathcal{Y}(\tau)$, $K(\tau)$, and $J(\tau)$ are illustrated in Figs. 4–7.

Note that now the order of all curves $\mathcal{Y}(\tau)$, $K(\tau)$, and $J(\tau)$ with respect to the parameter σ is the same, because the combination $(\sigma-1)$ enters Eq. (70) with the opposite sign, compared with the term $(\widetilde{\sigma}-1)$ in the similar equation (57).

5.3. Arbitrary Phases of Time-Dependent Coupling Coefficients

In the case of arbitrary phases ϕ_k of the coupling coefficients $\gamma_k(t)$ in the Hamiltonian (1), all formulas for the λ-matrices and, consequently, the elements of the covariance matrix and the squeezing and entanglement coefficients remain practically the same as those given above (see Appendix A). The only modification occurs in the definitions of the "slow time" dimensionless parameters $\tilde{\tau}$ and τ and phases $\widetilde{\varphi}$ and φ.

In the case of parametric amplifier ($\eta=\omega_1+\omega_2$), the most general definitions are as follows:

$$\tilde{\tau}=\varpi\widetilde{\kappa}t, \qquad \widetilde{\kappa}=\frac{1}{4}\left|\widetilde{\nu}-i\widetilde{\mu}\right|, \qquad (73)$$

$$\sin\widetilde{\varphi}=\frac{(\widetilde{\mu}+\widetilde{\mu}^*)+i(\widetilde{\nu}-\widetilde{\nu}^*)}{2\left|\widetilde{\nu}-i\widetilde{\mu}\right|}, \qquad \cos\widetilde{\varphi}=\frac{(\widetilde{\nu}+\widetilde{\nu}^*)-i(\widetilde{\mu}-\widetilde{\mu}^*)}{2\left|\widetilde{\nu}-i\widetilde{\mu}\right|}, \qquad (74)$$

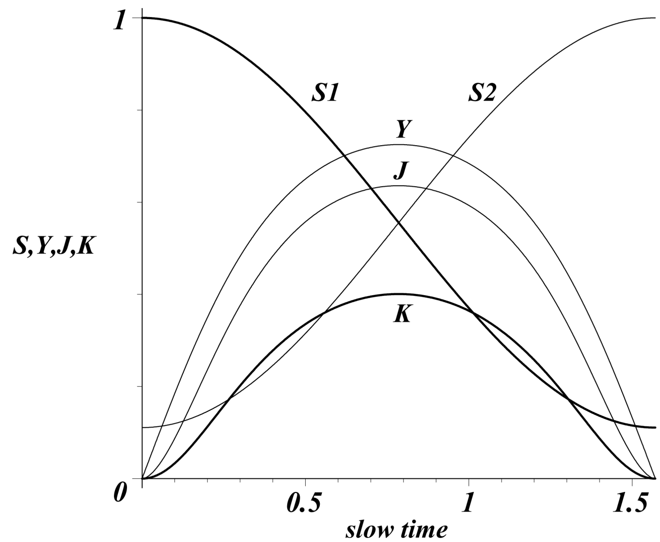

Fig. 4. The squeezing coefficients $S_1(\tau)$ and $S_2(\tau)$, the covariance entanglement coefficient $\mathcal{Y}(\tau)$, the "compact entropy" J (60), and the "linear entropy" K (34) versus the "slow time" τ in the case of a parametric converter for $\varrho_1 = 0$ and $\varrho_2 = 1.1$.

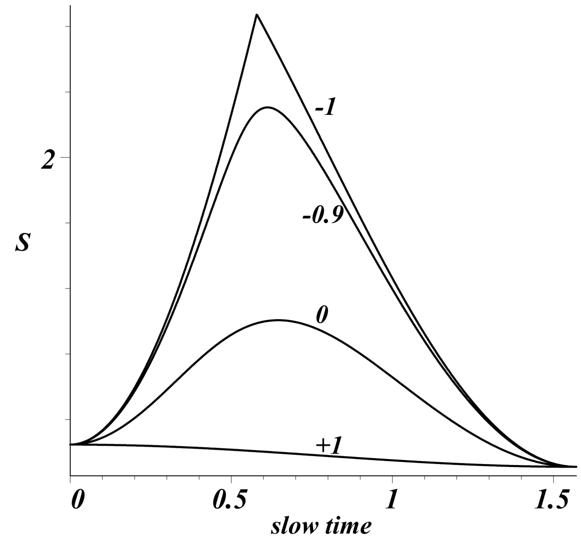

Fig. 5. The squeezing coefficient $S_1(\tau)$ versus the "slow time" τ in the case of a parametric converter for $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$. The numbers on the curves give the corresponding values of the parameter σ. The graphics for $S_2(\tau)$ can be obtained by means of the mirror reflection from the line $\tau = \pi/4$.

and

$$\widetilde{\mu} = \gamma_1 e^{-i\phi_1} - \gamma_4 e^{-i\phi_4}, \qquad \widetilde{\nu} = \gamma_2 e^{-i\phi_2} + \gamma_3 e^{-i\phi_3}. \tag{75}$$

In the case of parametric converter ($\eta = \omega_1 - \omega_2$), the following changes should be made:

$$\tau = \varpi \kappa t, \qquad \kappa = \frac{1}{4}\left|\nu + i\mu\right|, \tag{76}$$

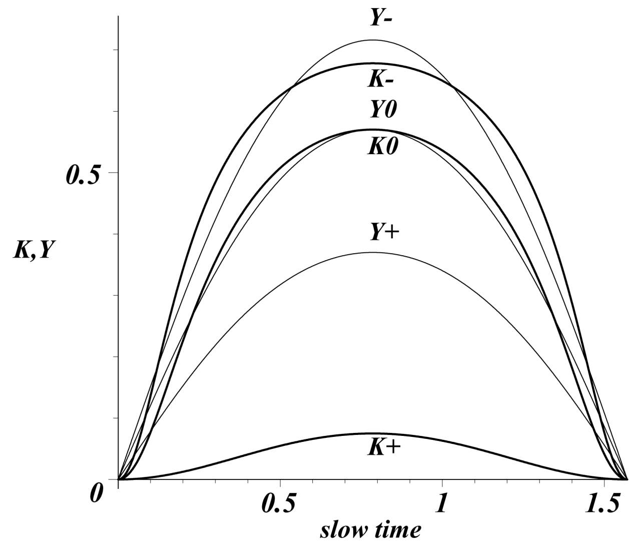

Fig. 6. The covariance entanglement coefficient $\mathcal{Y}$ (thin lines) and the linear entropy K (34) (thick lines) versus the "slow time" τ in the case of a parametric converter, respectively, for $\sigma = -1$ ($Y-$ and $K-$), $\sigma = 0$ ($Y0$ and $K0$), and $\sigma = 1$ ($Y+$ and $K+$). The initial squeezing parameters are $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$.

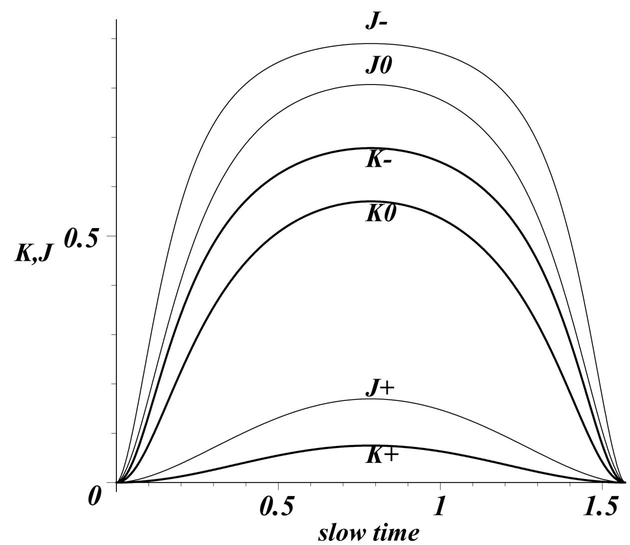

Fig. 7. The linear entropy $K_1 = K_2$ (34) and the "compact entropy" $J_1 = J_2$ (60) of initially pure Gaussian states versus the "slow time" τ in the case of a parametric converter, respectively, for $\sigma = -1$ ($K-$ and $J-$), $\sigma = 0$ ($K0$ and $J0$), and $\sigma = 1$ ($K+$ and $J+$). The initial squeezing parameters are $\varrho_1 = 0.7$ and $\varrho_2 = 1.1$.

$$\sin\varphi = \frac{(\mu+\mu^*) - i(\nu-\nu^*)}{2\,|\nu+i\mu|}, \qquad \cos\varphi = \frac{(\nu+\nu^*) + i(\mu-\mu^*)}{2\,|\nu+i\mu|}, \tag{77}$$

and

$$\mu = \gamma_1 e^{-i\phi_1} + \gamma_4 e^{-i\phi_4}, \qquad \nu = \gamma_2 e^{-i\phi_2} - \gamma_3 e^{-i\phi_3}. \tag{78}$$

The details are given in Appendix A.

6. Conclusion

The main results of our paper are as follows.

We have obtained explicit analytical solutions describing the behavior of two coupled quantum oscillators with the frequencies ω_1 and ω_2 for the most general weak bilinear resonance coupling, when the interaction terms are harmonic functions of time with arbitrary relative phases but with the same frequency $\eta = \omega_1 \pm \omega_2$. Using these solutions [which are expressed in terms of the elements of some symplectic matrices, belonging to the group $\mathrm{Sp}(4, R)$], we have constructed the time-dependent covariance matrices and studied the time evolution of certain combinations of the covariances, which are invariant with respect to free evolution of uncoupled oscillators, i.e., with respect to the group $\mathrm{Sp}(2, R) \otimes \mathrm{Sp}(2, R)$. Namely, the invariant combinations of the covariances related to the same mode give the squeezing coefficient, whereas the sum of squares of cross-covariances gives the entanglement coefficient. Although the measure of entanglement used is not as universal as the entropic measures for arbitrary states, it is much more simple from the viewpoint of calculations, especially for the dynamical problems where the covariances can be found by solving the Heisenberg equations of motion for the quadrature components, whereas the calculation of the timee-dependent density matrix requires much more efforts. Moreover, for the Gaussian states, our coefficient, which can vary in the interval $[0, 1)$, takes on zero value iff the state is completely disentangled. Therefore, it can provide useful information on the entanglement properties of the coupled continuous variable systems, as well as a new measure introduced recently in [65].

In the case of the parametric amplifier ($\eta = \omega_1 + \omega_2$), we have demonstrated that no bilinear coupling can squeeze the initial coherent state or improve the squeezing of the initial squeezed state. In this case, the minimum (with respect to the period of fast oscillations) values of quadrature component variances grow unlimitedly in the course of time. However, the entanglement coefficient tends to a finite value greater than $1/\sqrt{2}$ but less than some quantity $\mathcal{Y}_* < 1$, which is determined by the initial states of both modes and by the relations between the coupling coefficients.

In the case of parametric converter ($\eta = \omega_1 - \omega_2$), the modes periodically exchange their squeezing coefficients (as well as other characteristics — energy, purity, etc.). The period of such an exchange is inversely proportional to the strength of coupling. The maximum entanglement (which is reached for the first time in a quarter period of slow oscillations) depends on the relations between the coupling constants and the parameters of the initial states. In principle, it can assume values arbitrarily close to unity, but only for significantly different initial states — the absolute value of the difference of the initial squeezing parameters ϱ_1 and ϱ_2 [defined by means of Eqs. (18)–(20)] must be much greater than 1. On the contrary, if $\varrho_1 = \varrho_2$ then the maximum entanglement coefficient cannot exceed $1/\sqrt{2}$. Moreover, in this case, one can find a set of coupling constants that yields zero entanglement coefficient for all instants of time (in this case, there is no exchange of squeezing, since it is the same for both modes from the very beginning). In particular, coherent states ($\varrho_1 = \varrho_2 = 0$) always remain coherent in the process of parametric conversion (provided small corrections of the second order with respect to coupling constants can be neglected). This example shows how useful the entanglement coefficient can be for the analysis of the properties of coupled oscillators. In particular, it becomes clear why the harmonic oscillator, which was initially in a coherent state, remains in the (time-dependent) coherent state in the process of relaxation at zero temperature, described by means of the "standard master equation" [69–74]. Indeed, this equation is the consequence of the "microscopic" model of interaction between the oscillator involved and a large "heat bath" of harmonic oscillators. In the process of derivation, the coupling constants are assumed to be time independent and only resonance interactions with oscillators having the same frequency are taken into account while calculating the coefficients of the master equation. Besides, at zero temperature all the oscillators of the bath are in the ground state, which is a special case of coherent states.

In the case of parametric converter, different choices of the type of bilinear coupling can result in qualitatively different evolution pictures. For example, if $\vartheta_1 = \vartheta_2 = 0$ (both modes are initially squeezed in the x direction), then the coordinate–coordinate or momentum–momentum couplings yield $\varphi = \pm\pi/2$ and $\sigma = -1$, which leads to the nonmonotonous evolution of the squeezing coefficient, whereas the coordinate–momentum coupling yields $\varphi = 0$ or $\varphi = \pi$ and $\sigma = 1$, i.e., the monotonous evolution. In another extreme case where the initial directions of squeezing are perpendicular, we have $\vartheta_1 - \vartheta_2 = \pm\pi/2$. Then any particular coupling (x–x, p–p or x–p) results in $\sigma = 0$, and only specific combination of the x–x and x–p or p–p and x–p couplings can give the values $\sigma = \pm 1$. A similar sensitivity to the choice of coupling in the problem of quantum-state exchange was discovered in [35].

In this paper, we have considered only the cases of strict resonance. A small detuning can result in many additional interesting effects, as was shown for other (but similar) problems, e.g., in [75, 76]. Also, the problem of squeezing exchange and entanglement between several oscillators seems to be interesting. These subjects are under study and the results will be reported elsewhere.

Acknowledgments

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Appendix A. The Bogoliubov–Mitropolsky Scheme

The solutions of the differential equations for λ-matrices, such as, e.g., Eq. (41), can be obtained in an approximate form with the aid of the Bogoliubov–Mitropolsky averaging method [67]. We show the application of this method, calculating, for example, the elements of the matrix λ_1 which can be represented explicitly as follows:

$$\lambda_1 = \begin{bmatrix} v_1(t) & v_2(t) \\ v_3(t) & v_4(t) \end{bmatrix}. \tag{A.1}$$

Equation (41) connects only the elements of the same rows of matrix (A.1). Let us consider, for instance, the matrix elements $v_1(t)$ and $v_2(t)$, which satisfy the equations

$$\ddot{v}_1 + \omega_1^2 v_1 = h_1\dot{v}_2 - h_2 v_2 \equiv \varepsilon\mathcal{X}_1(v_2, \dot{v}_2, t), \tag{A.2}$$

$$\ddot{v}_2 + \omega_2^2 v_2 = h_3\dot{v}_1 - h_4 v_1 \equiv \varepsilon\mathcal{X}_2(v_1, \dot{v}_1, t), \tag{A.3}$$

where we have introduced a small parameter ε, supposing that all coupling constants γ_k are proportional to this parameter. The functions h_k, $k = 1, 2, 3, 4$, are given by Eqs. (43) and (44). At the initial time moment, we have $v_1(0) = 1$, $v_2(0) = 0$, whereas the initial values of the derivatives $\dot{v}_1(0)$ and $\dot{v}_2(0)$ can be found from Eqs. (37)–(40). If $\varepsilon = 0$, the system of equations (A.2)–(A.3) has an obvious solution:

$$v_1 = a_1 e^{i\omega_1 t} + a_{-1} e^{-i\omega_1 t}, \tag{A.4}$$

$$v_2 = a_2 e^{i\omega_2 t} + a_{-2} e^{-i\omega_2 t}, \tag{A.5}$$

where the constants a_k are determined by the initial conditions. Now, following [67] we reduce the system of equations (A.2)–(A.3) to a new system of first-order differential equations in the standard form. For this purpose, we introduce unknown complex conjugate functions of time $z_j(t)$ and $z_{-j}(t)$, $j = 1, 2$ as follows:

$$v_j = z_j e^{i\omega_j t} + z_{-j} e^{-i\omega_j t}, \tag{A.6}$$

$$\frac{dv_j}{dt} = i\omega_j z_j e^{i\omega_j t} - i\omega_j z_{-j} e^{-i\omega_j t}. \tag{A.7}$$

Differentiating (A.6) and comparing with (A.7) we see that the new functions z_k and z_{-k} must satisfy the differential equation

$$\frac{dz_j}{dt} e^{i\omega_j t} + \frac{dz_{-j}}{dt} e^{-i\omega_j t} = 0. \tag{A.8}$$

Writing all harmonic functions of time in the exponential form, such as

$$\gamma_i(t) = \frac{1}{2}\gamma_i \left[e^{i(\eta t - \phi_i)} + e^{-i(\eta t - \phi_i)} \right], \tag{A.9}$$

and excluding the first derivatives of functions $v_j(t)$ with the aid of Eq. (A.7), we can express the functions $\mathcal{X}_j$ on the right-hand sides of Eqs. (A.2) and (A.3) in terms of new functions $z_{\pm k}$ as follows,

$$\mathcal{X}_1 = \left[(i\omega_2 c_1^* - c_3^*)\, e^{i(\eta+\omega_2)t} + (i\omega_2 c_1 - c_3)\, e^{-i(\eta-\omega_2)t} \right] z_2 - \left[(i\omega_2 c_1^* + c_3^*)\, e^{i(\eta-\omega_2)t} + (i\omega_2 c_1 + c_3)\, e^{-i(\eta+\omega_2)t} \right] z_{-2}\,,$$

$$\mathcal{X}_2 = \left[(i\omega_1 c_5^* - c_7^*)\, e^{i(\eta+\omega_1)t} + (i\omega_1 c_5 - c_7)\, e^{-i(\eta-\omega_1)t} \right] z_1 - \left[(i\omega_1 c_5^* + c_7^*)\, e^{i(\eta-\omega_1)t} + (i\omega_1 c_5 + c_7)\, e^{-i(\eta+\omega_1)t} \right] z_{-1}\,,$$

where

$$\varepsilon c_1 = \frac{\varpi}{2\omega_2}\left[\left(\omega_2\gamma_2 e^{i\phi_2} - \omega_1\gamma_3 e^{i\phi_3}\right) - i\eta\gamma_1 e^{i\phi_1} \right], \qquad \varepsilon c_3 = \frac{\varpi}{2}\left[\left(\omega_1\gamma_4 e^{i\phi_4} + \omega_2\gamma_1 e^{i\phi_1}\right) + i\eta\gamma_2 e^{i\phi_2} \right],$$

$$\varepsilon c_5 = \frac{\varpi}{2\omega_1}\left[\left(\omega_1\gamma_3 e^{i\phi_3} - \omega_2\gamma_2 e^{i\phi_2}\right) - i\eta\gamma_1 e^{i\phi_1} \right], \qquad \varepsilon c_7 = \frac{\varpi}{2}\left[\left(\omega_1\gamma_1 e^{i\phi_1} + \omega_2\gamma_4 e^{i\phi_4}\right) + i\eta\gamma_3 e^{i\phi_3} \right].$$

Differentiating Eq. (A.7) and replacing the second derivatives by their expressions in Eqs. (A.2) and (A.3), we arrive at the equations

$$i\omega_j \dot{z}_j e^{i\omega_j t} - i\omega_j \dot{z}_{-j} e^{-i\omega_j t} = \varepsilon \mathcal{X}_j(z,t), \qquad j = 1, 2. \tag{A.10}$$

Combining the differential equations (A.8) and (A.10) we obtain for the functions $z_k(t)$ a set of first-order equations:

$$\dot{z}_1 = \frac{\varepsilon}{2i\omega_1}\mathcal{X}_1 e^{-i\omega_1 t}, \qquad \dot{z}_{-1} = -\frac{\varepsilon}{2i\omega_1}\mathcal{X}_1 e^{i\omega_1 t}, \tag{A.11}$$

$$\dot{z}_2 = \frac{\varepsilon}{2i\omega_2}\mathcal{X}_2 e^{-i\omega_2 t}, \qquad \dot{z}_{-2} = -\frac{\varepsilon}{2i\omega_2}\mathcal{X}_2 e^{i\omega_2 t}, \tag{A.12}$$

which can be represented in the following standard form:

$$\frac{dz_j}{dt} = \varepsilon \mathcal{G}_j(t,z), \qquad j = \pm 1, \pm 2, \tag{A.13}$$

$$\mathcal{G}_1 = \mathcal{G}^*_{-1} = \frac{1}{2\omega_1}\left\{\left[(\omega_2 c_1^* + ic_3^*)\, e^{i(\eta-\omega_1+\omega_2)t} + (\omega_2 c_1 + ic_3)\, e^{-i(\eta+\omega_1-\omega_2)t}\right] z_2 - \left[(\omega_2 c_1^* - ic_3^*)\, e^{i(\eta-\omega_1-\omega_2)t} + (\omega_2 c_1 - ic_3)\, e^{-i(\eta+\omega_1+\omega_2)t}\right] z_{-2}\right\}, \tag{A.14}$$

$$\mathcal{G}_2 = \mathcal{G}^*_{-2} = \frac{1}{2\omega_2}\left\{\left[(\omega_1 c_5^* + ic_7^*)\, e^{i(\eta+\omega_1-\omega_2)t} + (\omega_1 c_5 + ic_7)\, e^{-i(\eta-\omega_1+\omega_2)t}\right] z_1 - \left[(\omega_1 c_5^* - ic_7^*)\, e^{i(\eta-\omega_1-\omega_2)t} + (\omega_1 c_5 - ic_7)\, e^{-i(\eta+\omega_1+\omega_2)t}\right] z_{-1}\right\}. \tag{A.15}$$

According to [67], the solutions to the set of equations (A.13) can be represented in the form of an asymptotic series with respect to the small parameter ε

$$z_j(t) = \xi_j^{(0)}(t) + \varepsilon \xi_j^{(1)}(t) + \varepsilon^2 \xi_j^{(2)}(t) + \cdots ,$$

where the first terms $\xi_j^{(0)}$ satisfy the set of differential equations with constant coefficients, which can be obtained by averaging Eq. (A.13) in time:

$$\frac{d\xi_j^{(0)}}{dt} = \lim_{T\to\infty} \frac{\varepsilon}{T} \int_0^T \mathcal{G}_j(t', \xi) dt'. \tag{A.16}$$

Since we are not interested here in corrections to solutions of the order of ε, we shall omit the supescript (0), writing hereafter simply $\xi_j(t)$ instead of $\xi_j^{(0)}(t)$. The initial conditions for the functions $\xi_{\pm j}$ can be easily derived from the initial conditions for the functions v_j:

$$\xi_j(0) = \frac{1}{2i\omega_j}\left[i\omega_j v_j(0) + \frac{dv_j(0)}{dt}\right], \qquad \xi_{-j}(0) = \frac{1}{2i\omega_j}\left[i\omega_j v_j(0) - \frac{dv_j(0)}{dt}\right].$$

Evidently, for the functions (A.14) and (A.15), the limit value of the integral on the right-hand side of (A.16) differs from zero only under the conditions $\eta = \pm\omega_1 \pm \omega_2$.[2]

Consider, for example, the case $\eta = \omega_1 - \omega_2$ (parametric converter). Then only one term survives after averaging the right-hand sides of Eqs. (A.14) and (A.15), namely, the first term in (A.14) and the second term in (A.15). Taking into account the explicit form of the coefficients c_k, we arrive at the following equations for slow varying amplitudes ξ_j (and complex conjugated equations for the functions ξ_{-j}):

$$\frac{d\xi_1}{dt} = \frac{\varpi}{4}(\nu + i\mu)\xi_2, \qquad \frac{d\xi_2}{dt} = -\frac{\varpi}{4}(\nu^* - i\mu^*)\xi_1, \tag{A.17}$$

where the coefficients μ and ν are given by Eq. (78). Looking for the solutions in the form $\xi_j(t) \sim \exp(\lambda t)$, one can easily find that

$$\lambda^2 = -\frac{\varpi^2}{16}\,|\nu + i\mu|^2 = -\varpi^2\kappa^2.$$

Consequently, the solutions have oscillating character, being expressed in terms of trigonometric functions of the "slow time" τ defined in Eq. (76).

[2]As a matter a fact, a slightly modified scheme works under a weaker restriction, e.g., $|\eta \pm \omega_1 \pm \omega_2| \ll \sqrt{\omega_1\omega_2}$ (a small detuning from the resonance), but in this paper we confine ourselves to the cases of strict resonance.

Note, however, that in the special case $\eta = \omega_1 - \omega_2 = 0$, *two* first terms on the right-hand sides of Eqs. (A.14) and (A.15) do not depend on time and, consequently, survive after averaging. In this case, instead of (A.17) we have

$$\frac{d\xi_1}{dt} = \frac{\varpi}{2}(\mathrm{Re}\,\nu + i\mathrm{Re}\,\mu)\xi_2, \qquad \frac{d\xi_2}{dt} = -\frac{\varpi}{2}(\mathrm{Re}\,\nu - i\mathrm{Re}\,\mu)\xi_1, \tag{A.18}$$

so that

$$\lambda^2 = -\frac{\varpi^2}{4}\left[(\mathrm{Re}\,\nu)^2 + (\mathrm{Re}\,\mu)^2\right] = -\varpi^2\kappa_0^2.$$

If $\phi_j \equiv 0$, then $|\kappa_0| = 2|\kappa|$, as was mentioned in the main text. However, for arbitrary values of ϕ_j, the situation may be different.

Equations (A.17) can be justified if the terms proportional to $\exp(\pm 2i\eta t)$ [arising after putting $\eta = \omega_1 - \omega_2$ in the functions (A.14) and (A.15)] "die out" after averaging over many periods of "fast" oscillations (with frequencies ω_1 or ω_2). On the other hand, the averaging time cannot exceed $(\varpi\kappa)^{-1}$, because the slow time $\tau = \varpi\kappa t$ should not change appreciably during averaging. Consequently, Eqs. (A.17) are valid provided $\eta \gg \varpi\kappa$. A direct transition from the case $\omega_1 - \omega_2 \neq 0$ to the case $\omega_1 - \omega_2 = 0$ is impossible within the framework of the simplest version of the Bogoliubov–Mitropolsky scheme used here. This requires a special study, which is not considered here.

In the case $\eta = \omega_1 + \omega_2$ (parametric amplifier), only the third terms in Eqs. (A.14) and (A.15) survive after averaging, resulting in the equations (remember that $z_j^* = z_{-j}$)

$$\frac{d\xi_1}{dt} = \frac{\varpi}{4}(\widetilde{\nu} - i\widetilde{\mu})\xi_2^*, \qquad \frac{d\xi_2^*}{dt} = \frac{\varpi}{4}(\widetilde{\nu}^* + i\widetilde{\mu}^*)\xi_1, \tag{A.19}$$

where the coefficients $\widetilde{\mu}$ and $\widetilde{\nu}$ are given by Eq. (75). Looking for solutions proportional to $\exp(\lambda t)$, we obtain

$$\lambda^2 = \frac{\varpi^2}{16}\,|\widetilde{\nu} - i\widetilde{\mu}|^2 = \varpi^2\widetilde{\kappa}^2,$$

which means that the solutions are expressed in terms of hyperbolic functions of the "slow time" $\tilde{\tau}$ defined in Eq. (73).

Appendix B. Elements of λ-Matrices and Covariance Matrices

The further procedure is quite straightforward — it mainly consists of solving linear algebraic equations arising from the initial conditions. Omitting the details of cumbersome calculations, we give here the explicit forms of elements of matrices λ_j in the most general case (i.e., for different phases ϕ_j).

For the parametric amplifier,

$$\begin{array}{ll} \lambda_1^{11} = \cosh\tilde\tau\cos\omega_1 t, & \lambda_1^{12} = \sinh\tilde\tau\cos(\omega_2 t - \widetilde\varphi), \\ \lambda_1^{21} = \sinh\tilde\tau\cos(\omega_1 t - \widetilde\varphi), & \lambda_1^{22} = \cosh\tilde\tau\cos\omega_2 t, \\ \lambda_2^{11} = \cosh\tilde\tau\sin\omega_1 t, & \lambda_2^{12} = \sinh\tilde\tau\sin(\omega_2 t - \widetilde\varphi), \\ \lambda_2^{21} = \sinh\tilde\tau\sin(\omega_1 t - \widetilde\varphi), & \lambda_2^{22} = \cosh\tilde\tau\sin\omega_2 t, \\ \lambda_3^{11} = -\cosh\tilde\tau\sin\omega_1 t, & \lambda_3^{12} = \sinh\tilde\tau\sin(\omega_2 t - \widetilde\varphi), \\ \lambda_3^{21} = \sinh\tilde\tau\sin(\omega_1 t - \widetilde\varphi), & \lambda_3^{22} = -\cosh\tilde\tau\sin\omega_2 t, \\ \lambda_4^{11} = \cosh\tilde\tau\cos\omega_1 t, & \lambda_4^{12} = -\sinh\tilde\tau\cos(\omega_2 t - \widetilde\varphi), \\ \lambda_4^{21} = -\sinh\tilde\tau\cos(\omega_1 t - \widetilde\varphi), & \lambda_4^{22} = \cosh\tilde\tau\cos\omega_2 t, \end{array} \tag{B.1}$$

where $\widetilde\varphi$ and $\tilde\tau$ are given in Eqs. (51) and (52) or Eqs. (73)–(75).

For the parametric converter,

$$\begin{array}{ll} \lambda_1^{11} = \cos\tau\cos\omega_1 t, & \lambda_1^{12} = -\sin\tau\cos(\omega_2 t - \varphi), \\ \lambda_1^{21} = \sin\tau\cos(\omega_1 t + \varphi), & \lambda_1^{22} = \cos\tau\cos\omega_2 t, \\ \lambda_2^{11} = \cos\tau\sin\omega_1 t, & \lambda_2^{12} = -\sin\tau\cos(\omega_2 t - \varphi), \\ \lambda_2^{21} = \sin\tau\sin(\omega_1 t + \varphi), & \lambda_2^{22} = \cos\tau\sin\omega_2 t, \\ \lambda_3^{11} = -\cos\tau\sin\omega_1 t, & \lambda_3^{12} = \sin\tau\sin(\omega_2 t - \varphi), \\ \lambda_3^{21} = -\sin\tau\sin(\omega_1 t + \varphi), & \lambda_3^{22} = -\cos\tau\sin\omega_2 t, \\ \lambda_4^{11} = \cos\tau\cos\omega_1 t, & \lambda_4^{12} = -\sin\tau\cos(\omega_2 t - \varphi), \\ \lambda_4^{21} = \sin\tau\cos(\omega_1 t + \varphi), & \lambda_4^{22} = \cos\tau\cos\omega_2 t, \end{array} \tag{B.2}$$

where φ and τ are given in Eqs. (61) and (62) or Eqs. (76)–(78).

The matrices whose elements are given above satisfy the initial conditions and the symplectic conditions (13). Calculating the covariance matrix with the aid of Eqs. (9), (12), and (17), it is convenient to separate the "short-time" and "long-time" dependences.

In the case of parametric amplifier, we write

$$\widetilde{\mathcal{M}}(t,\tilde\tau) = \widetilde{\mathcal{A}}(t)\sinh^2\tilde\tau + \widetilde{\mathcal{B}}(t)\cosh^2\tilde\tau + \widetilde{\mathcal{C}}(t)\sinh\tilde\tau\cosh\tilde\tau, \tag{B.3}$$

where the nonzero elements of the symmetric matrices $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{C}}$ are as follows:

$$\begin{aligned} \widetilde{\mathcal{A}}_{p_1p_1} &= \frac{1}{2}\Big[\mathcal{P}_2\cos^2(\omega_1 t - \widetilde\varphi) + \mathcal{X}_2\sin^2(\omega_1 t - \widetilde\varphi) + \mathcal{R}_2\sin(2\omega_1 t - 2\widetilde\varphi)\Big], \\ \widetilde{\mathcal{A}}_{p_1x_1} &= \frac{1}{4}\Big[(\mathcal{P}_2 - \mathcal{X}_2)\sin(2\omega_1 t - 2\widetilde\varphi) - 2\mathcal{R}_2\cos(2\omega_1 t - 2\widetilde\varphi)\Big], \\ \widetilde{\mathcal{A}}_{p_2p_2} &= \frac{1}{2}\Big[\mathcal{P}_1\cos^2(\omega_2 t - \widetilde\varphi) + \mathcal{X}_1\sin^2(\omega_2 t - \widetilde\varphi) + \mathcal{R}_1\sin(2\omega_2 t - 2\widetilde\varphi)\Big], \\ \widetilde{\mathcal{A}}_{p_2x_2} &= \frac{1}{4}\Big[(\mathcal{P}_1 - \mathcal{X}_1)\sin(2\omega_2 t - 2\widetilde\varphi) - 2\mathcal{R}_1\cos(2\omega_2 t - 2\widetilde\varphi)\Big], \\ \widetilde{\mathcal{A}}_{x_1x_1} &= \frac{1}{2}\Big[\mathcal{P}_2\sin^2(\omega_1 t - \widetilde\varphi) + \mathcal{X}_2\cos^2(\omega_1 t - \widetilde\varphi) - \mathcal{R}_2\sin(2\omega_1 t - 2\widetilde\varphi)\Big], \\ \widetilde{\mathcal{A}}_{x_2x_2} &= \frac{1}{2}\Big[\mathcal{P}_1\sin^2(\omega_2 t - \widetilde\varphi) + \mathcal{X}_1\cos^2(\omega_2 t - \widetilde\varphi) - \mathcal{R}_1\sin(2\omega_2 t - 2\widetilde\varphi)\Big], \end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{B}}_{p_1p_1} &= \frac{1}{2}\Big[\mathcal{P}_1\cos^2\omega_1 t+\mathcal{X}_1\sin^2\omega_1 t-\mathcal{R}_1\sin 2\omega_1 t\Big],\\
\widetilde{\mathcal{B}}_{p_1x_1} &= \frac{1}{4}\Big[(\mathcal{P}_1-\mathcal{X}_1)\sin 2\omega_1 t+2\mathcal{R}_1\cos 2\omega_1 t\Big],\\
\widetilde{\mathcal{B}}_{p_2p_2} &= \frac{1}{2}\Big[\mathcal{P}_2\cos^2\omega_2 t+\mathcal{X}_2\sin^2\omega_2 t-\mathcal{R}_2\sin 2\omega_2 t\Big],\\
\widetilde{\mathcal{B}}_{p_2x_2} &= \frac{1}{4}\Big[(\mathcal{P}_2-\mathcal{X}_2)\sin 2\omega_2 t+2\mathcal{R}_2\cos 2\omega_2 t\Big],\\
\widetilde{\mathcal{B}}_{x_1x_1} &= \frac{1}{2}\Big[\mathcal{P}_1\sin^2\omega_1 t+\mathcal{X}_1\cos^2\omega_1 t+\mathcal{R}_1\sin 2\omega_1 t\Big],\\
\widetilde{\mathcal{B}}_{x_2x_2} &= \frac{1}{2}\Big[\mathcal{P}_2\sin^2\omega_2 t+\mathcal{X}_2\cos^2\omega_2 t+\mathcal{R}_2\sin 2\omega_2 t\Big],
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{C}}_{p_1p_2} &= -\frac{1}{2}\Big[\mathcal{P}_2\cos\omega_2 t\cos(\omega_1 t-\widetilde{\varphi})-\mathcal{X}_2\sin\omega_2 t\sin(\omega_1 t-\widetilde{\varphi})+\mathcal{R}_2\sin(\omega_- t-\widetilde{\varphi})\\
&\quad+\mathcal{P}_1\cos\omega_1 t\cos(\omega_2 t-\widetilde{\varphi})-\mathcal{X}_1\sin\omega_1 t\sin(\omega_2 t-\widetilde{\varphi})-\mathcal{R}_1\sin(\omega_- t+\widetilde{\varphi})\Big],\\
\widetilde{\mathcal{C}}_{p_1x_2} &= -\frac{1}{2}\Big[\mathcal{P}_2\sin\omega_2 t\cos(\omega_1 t-\widetilde{\varphi})+\mathcal{X}_2\cos\omega_2 t\sin(\omega_1 t-\widetilde{\varphi})+\mathcal{R}_2\cos(\omega_- t-\widetilde{\varphi})\\
&\quad+\mathcal{P}_1\cos\omega_1 t\sin(\omega_2 t-\widetilde{\varphi})+\mathcal{X}_1\sin\omega_1 t\cos(\omega_2 t-\widetilde{\varphi})-\mathcal{R}_1\cos(\omega_- t+\widetilde{\varphi})\Big],\\
\widetilde{\mathcal{C}}_{p_2x_1} &= -\frac{1}{2}\Big[\mathcal{P}_2\cos\omega_2 t\sin(\omega_1 t-\widetilde{\varphi})+\mathcal{X}_2\sin\omega_2 t\cos(\omega_1 t-\widetilde{\varphi})-\mathcal{R}_2\cos(\omega_- t-\widetilde{\varphi})\\
&\quad+\mathcal{P}_1\sin\omega_1 t\cos(\omega_2 t-\widetilde{\varphi})+\mathcal{X}_1\cos\omega_1 t\sin(\omega_2 t-\widetilde{\varphi})+\mathcal{R}_1\cos(\omega_- t+\widetilde{\varphi})\Big],\\
\widetilde{\mathcal{C}}_{x_1x_2} &= -\frac{1}{2}\Big[\mathcal{P}_2\sin\omega_2 t\sin(\omega_1 t-\widetilde{\varphi})-\mathcal{X}_2\cos\omega_2 t\cos(\omega_1 t-\widetilde{\varphi})+\mathcal{R}_2\sin(\omega_- t-\widetilde{\varphi})\\
&\quad+\mathcal{P}_1\sin\omega_1 t\sin(\omega_2 t-\widetilde{\varphi})-\mathcal{X}_1\cos\omega_1 t\cos(\omega_2 t-\widetilde{\varphi})-\mathcal{R}_1\sin(\omega_- t+\widetilde{\varphi})\Big],
\end{aligned}$$

with $\omega_-=\omega_1-\omega_2$.

In the case of parametric converter, we write the covariance matrix in the form

$$\mathcal{M}(t,\tau)=\mathcal{A}(t)\sin^2\tau+\mathcal{B}(t)\cos^2\tau+\mathcal{C}(t)\sin\tau\cos\tau. \tag{B.4}$$

The nonzero elements of matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are as follows:

$$\begin{aligned}
\mathcal{A}_{p_1p_1} &= \frac{1}{2}\Big[\mathcal{P}_2\cos^2(\omega_1 t+\varphi)+\mathcal{X}_2\sin^2(\omega_1 t+\varphi)-\mathcal{R}_2\sin(2\omega_1 t+2\varphi)\Big],\\
\mathcal{A}_{p_1x_1} &= \frac{1}{4}\Big[(\mathcal{P}_2-\mathcal{X}_2)\sin(2\omega_1 t+2\varphi)+2\mathcal{R}_2\cos(2\omega_1 t+2\varphi)\Big],\\
\mathcal{A}_{p_2p_2} &= \frac{1}{2}\Big[\mathcal{P}_1\cos^2(\omega_2 t-\varphi)+\mathcal{X}_1\sin^2(\omega_2 t-\varphi)-\mathcal{R}_1\sin(2\omega_2 t-2\varphi)\Big],\\
\mathcal{A}_{p_2x_2} &= \frac{1}{4}\Big[(\mathcal{P}_1-\mathcal{X}_1)\sin(2\omega_2 t-2\varphi)+2\mathcal{R}_1\cos(2\omega_2 t-2\varphi)\Big],\\
\mathcal{A}_{x_1x_1} &= \frac{1}{2}\Big[\mathcal{P}_2\sin^2(\omega_1 t+\varphi)+\mathcal{X}_2\cos^2(\omega_1 t+\varphi)+\mathcal{R}_2\sin(2\omega_1 t+2\varphi)\Big],\\
\mathcal{A}_{x_2x_2} &= \frac{1}{2}\Big[\mathcal{P}_1\sin^2(\omega_2 t-\varphi)+\mathcal{X}_1\cos^2(\omega_2 t-\varphi)+\mathcal{R}_1\sin(2\omega_2 t-2\varphi)\Big],
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_{p_1p_1} &= \frac{1}{2}\Big[\mathcal{P}_1\cos^2\omega_1 t + \mathcal{X}_1\sin^2\omega_1 t - \mathcal{R}_1\sin 2\omega_1 t\Big],\\
\mathcal{B}_{p_1x_1} &= \frac{1}{4}\Big[(\mathcal{P}_1 - \mathcal{X}_1)\sin 2\omega_1 t + 2\mathcal{R}_1\cos 2\omega_1 t\Big],\\
\mathcal{B}_{p_2p_2} &= \frac{1}{2}\Big[\mathcal{P}_2\cos^2\omega_2 t + \mathcal{X}_2\sin^2\omega_2 t - \mathcal{R}_2\sin 2\omega_2 t\Big],\\
\mathcal{B}_{p_2x_2} &= \frac{1}{4}\Big[(\mathcal{P}_2 - \mathcal{X}_2)\sin 2\omega_2 t + 2\mathcal{R}_2\cos 2\omega_2 t\Big],\\
\mathcal{B}_{x_1x_1} &= \frac{1}{2}\Big[\mathcal{P}_1\sin^2\omega_1 t + \mathcal{X}_1\cos^2\omega_1 t + \mathcal{R}_1\sin 2\omega_1 t\Big],\\
\mathcal{B}_{x_2x_2} &= \frac{1}{2}\Big[\mathcal{P}_2\sin^2\omega_2 t + \mathcal{X}_2\cos^2\omega_2 t + \mathcal{R}_2\sin 2\omega_2 t\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_{p_1p_2} &= \frac{1}{2}\Big[\mathcal{P}_2\cos\omega_2 t\cos(\omega_1 t+\varphi) + \mathcal{X}_2\sin\omega_2 t\sin(\omega_1 t+\varphi) - \mathcal{R}_2\sin(\omega_+ t+\varphi)\\
&\quad -\mathcal{P}_1\cos\omega_1 t\cos(\omega_2 t-\varphi) - \mathcal{X}_1\sin\omega_1 t\sin(\omega_2 t-\varphi) + \mathcal{R}_1\sin(\omega_+ t-\varphi)\Big],\\
\mathcal{C}_{p_1x_2} &= \frac{1}{2}\Big[\mathcal{P}_2\sin\omega_2 t\cos(\omega_1 t+\varphi) - \mathcal{X}_2\cos\omega_2 t\sin(\omega_1 t+\varphi) + \mathcal{R}_2\cos(\omega_+ t+\varphi)\\
&\quad -\mathcal{P}_1\cos\omega_1 t\sin(\omega_2 t-\varphi) + \mathcal{X}_1\sin\omega_1 t\cos(\omega_2 t-\varphi) - \mathcal{R}_1\cos(\omega_+ t-\varphi)\Big],\\
\mathcal{C}_{p_2x_1} &= \frac{1}{2}\Big[\mathcal{P}_2\cos\omega_2 t\sin(\omega_1 t+\varphi) - \mathcal{X}_2\sin\omega_2 t\cos(\omega_1 t+\varphi) + \mathcal{R}_2\cos(\omega_+ t+\varphi)\\
&\quad -\mathcal{P}_1\sin\omega_1 t\cos(\omega_2 t-\varphi) + \mathcal{X}_1\cos\omega_1 t\sin(\omega_2 t-\varphi) - \mathcal{R}_1\cos(\omega_+ t-\varphi)\Big],\\
\mathcal{C}_{x_1x_2} &= \frac{1}{2}\Big[\mathcal{P}_2\sin\omega_2 t\sin(\omega_1 t+\varphi) + \mathcal{X}_2\cos\omega_2 t\cos(\omega_1 t+\varphi) + \mathcal{R}_2\sin(\omega_+ t+\varphi)\\
&\quad -\mathcal{P}_1\sin\omega_1 t\sin(\omega_2 t-\varphi) - \mathcal{X}_1\cos\omega_1 t\cos(\omega_2 t-\varphi) - \mathcal{R}_1\sin(\omega_+ t-\varphi)\Big],
\end{aligned}
$$

with $\omega_+ = \omega_1 + \omega_2$.

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