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The phase state revisited: the Heisenberg limit in a quantum nondemolition measurement and the nonclassical depth of the state

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Abstract

We consider a quantum nondemolition measurement, the apparatus consisting of a Mach–Zehnder single-photon (signal) interferometer where in one arm the signal interacts (through a Kerr medium) with a probe beam. This interaction affects the visibility of the interferometer and induces a phase shift in the state of the probe beam. Here we investigate the effects of such interaction when the probe state is in the truncated phase state, $|\Theta_m\rangle = (1+N)^{-1/2}\sum_{n=0}^{N} e^{in\Theta_m}|n\rangle$, analyze the Heisenberg limit for the phase shift measurement, and its possible relation to the nonclassical depth of the state. © 2001 Published by Elsevier Science B.V.

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The phase problem in quantum optics is an old story initiated by Dirac [1] which introduced the commutation relation $[\hat{n}, \hat{\phi}] = i$, leading to the Heisenberg inequality $\Delta\hat{n}\Delta\hat{\phi} \geqslant 1$. Here $\hat{n}$ ($\hat{\phi}$) is the number (phase) operator and $\Delta\hat{n}$ ($\Delta\hat{\phi}$) characterizes the number (phase) uncertainty. A lengthy discussion on that relation, mainly concerning with what would be a good definition of a (Hermitian) phase operator, has been reported in many papers [2]. Although not being consensual [3], the problem now seems to be well addressed by a proposal of Pegg and Barnett [4], where satisfactory phase state and phase operator have been introduced.

The study about optimization of the phase of a general field state [5] and its determination [6], as well as the generation of a phase state [7] and its nonclassical depth [8] are subjects present in the recent literature. In this Letter we will take advantage of previous studies [9,10], to discuss the Heisenberg limit (HL) $\Delta\hat{\phi} \approx 1/\bar{n}$, which relates the limit on the sensitivity in a precision phase measurement to the available mean number of photons in the field of a *truncated phase state* and the possible relation to its classical depth. The HL will be established by treating on single photon interference plus a quantum nondemolition [11] scheme measurement, in order to relate the visibility of the interference pattern to the

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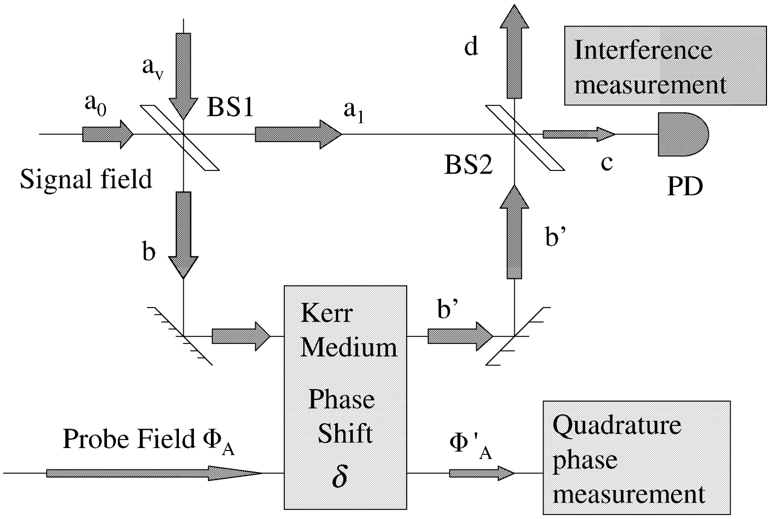

Fig. 1. Quantum nondemolition measurement apparatus. A single photon signal (a_0) goes into one port of a Mach–Zehnder interferometer while vacuum (a_v) goes through the other entrance. A 50% : 50% beam splitter (BS1) permits the photon to travel either through one arm (a_1) or/and through the other (b, b'); following path (b, b') it will interact, via Kerr effect, with a probe beam Φ_A in a nonlinear crystal. After the interaction the probe field Φ'_A contains an extra phase, δ, which can be measured at the quadrature phase measurement apparatus, permitting determination of the signal-to-noise ratio of the signal field (particle-like quality). At the exit port c the photon detector (PD) measures the photon number giving information on interference pattern (visibility) of the interferometer (wave-like quality).

phase shift built through a Kerr medium in a truncated phase state probe.

Fig. 1 shows a schematic Mach–Zehnder (MZ) interferometer for a single photon signal, where in one arm the field interacts, nonlinearly, with a probe field Φ_A through a Kerr medium [12]; BS1 and BS2 are 50% : 50% beam splitters and $a_0, a_v, a_1, b, b', c, d$, stand for the annihilation operators associated to the fields involved: a_0 (input signal), a_v (input vacuum), c and d are for the output fields and a_1, b, b' for the split beams. A phase shift δ is introduced in one arm of the apparatus, produced by the Kerr medium. Accordingly, the accurate phase sensitive measurement in the phase of the probe field Φ'_A destroys the interference pattern generated in the Mach–Zehnder apparatus. At the output of the interferometer the state of both fields (signal and probe) is given by

$$|\Psi_{\text{out}}\rangle = \frac{1}{2}\Big\{\big(1 - e^{i\delta\hat{A}^\dagger\hat{A}}\big)|1_d 0_c\rangle + i\big(1 + e^{i\delta\hat{A}^\dagger\hat{A}}\big)|0_d 1_c\rangle\Big\} \otimes |\Phi_A\rangle \tag{1}$$

(a superposition for the photon emerging from ports c and d), showing that the probability amplitude for each ports depends on the state of the probe field *and* on the phase δ. The operators $\hat{A}^\dagger$ (photon creation) and $\hat{A}$ (photon annihilation) act on the probe state $|\Phi_A\rangle$ and $e^{i\delta\hat{A}^\dagger\hat{A}}$ is a unitary operator standing for the action of the Kerr medium [13] upon the state evolution; we denote $|\Phi'_A\rangle \equiv e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi_A\rangle$ the probe state after the interaction is terminated. Measurement of the phase δ on the probe beam provides information about the photon number, whose exact knowledge in one path of the single-photon interferometer determines which path the photon follows, thus destroying the figure of interference. So, the accuracy of the phase measurement on the probe field influences the result of the interference. This can be quantified by the *visibility* as follows: the probability for the signal photon being detected at either output ports, c or d, is given by

$$P_{\binom{c}{d}} = \frac{1}{2}\big\|\big(1 \pm e^{i\delta\hat{A}^\dagger\hat{A}}\big)|\Phi_A\rangle\big\|^2 = 1 \pm \cos\theta\big|\langle\Phi_A|\Phi_{A'}\rangle\big|, \tag{2}$$

where the angle θ is defined as

$$e^{i\theta} \equiv \frac{\langle\Phi_A|\Phi_{A'}\rangle}{|\langle\Phi_A|\Phi_{A'}\rangle|} = \frac{\langle\Phi_A|e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi_A\rangle}{|\langle\Phi_A|e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi_A\rangle|}. \tag{3}$$

Since the maximum and minimum values of probabilities (2) are $P^{\max} = 1 + |\langle\Phi_A|\Phi_{A'}\rangle|$ and $P^{\min} = 1 - |\langle\Phi_A|\Phi_{A'}\rangle|$, the *visibility* of the interference pattern is given by

$$v = \frac{P^{\max} - P^{\min}}{P^{\max} + P^{\min}} = \left|\langle\Phi_A|\Phi'_A\rangle\right|. \tag{4}$$

When we are able to tell "which path" (a_1 or b in Fig. 1) the photon has followed, then it is possible to resolve the phase δ and according to the complementarity principle, the figure of interference in the interferometer is washed out, namely, $v \to 0$.

Now, as a preliminary, let us assume a probe field being in a Fock state, i.e., $|\Phi_A\rangle = |N\rangle$. In this case from Eq. (4) we obtain

$$v = \left|\langle N|e^{i\delta\hat{A}^\dagger\hat{A}}|N\rangle\right| = 1, \tag{5}$$

and no destruction of interference occurs in the interferometer, meaning that we cannot determine the phase shift δ using a field being in a number state. This result is in agreement with a general proof given in [10] that a high-precision phase measurement requires a (high-intensity) probe field being very super-Poissonian, i.e., $\Delta\hat{n}^2 \gg \bar{n}$ [14]. This is exactly the very reverse property of a number state, which is maximally sub-Poissonian ($\Delta\hat{n}^2 = 0$).

Next, let us assume the opposite situation: the probe field is a truncated phase state, i.e., $|\Phi_A\rangle = |\Theta_m\rangle$, where [4]

$$|\Theta_m\rangle = (1+N)^{-1/2}\sum_{n=0}^{N} e^{in\Theta_m}|n\rangle \tag{6}$$

(N finite) with $\Theta_m = \Theta_0 + 2\pi m/(N+1)$, $m = 0, 1, 2, \ldots, N$, Θ_0 being a reference phase for the state. Calculating the visibility, we now obtain

$$v = \left|\langle\Theta_m|e^{i\delta\hat{A}^\dagger\hat{A}}|\Theta_m\rangle\right| = (1+N)^{-1}\left|\sum_{n=0}^{N} e^{in\delta}\right| = \frac{1}{N+1}\left|\frac{\sin((N+1)\delta/2)}{\sin(\delta/2)}\right|, \tag{7}$$

and since the mean number of quanta in state (6) is $\bar{n} = N/2$, then for $\delta \ll 1$ and $N \gg 1$ we may write Eq. (7) as (with $\sin(\delta/2) \simeq \delta/2$ and $N+1 \simeq N$)

$$v \simeq \frac{|\sin(\bar{n}\delta)|}{\bar{n}\delta}. \tag{8}$$

Since $v < 1/(\bar{n}\delta)$, the visibility $v \to 0$ when $\bar{n}\delta \gg 1$, irrespective of the numerator value (even for $\bar{n}\delta = \pi$, when $v = 0$, the inequality $\delta > 1/\bar{n}$ holds). Thus, the smaller phase δ to be measured is limited by the quantity N of number of states that contribute in Eq. (6), and for a very small δ the HL $\delta \approx 1/\bar{n}$ may always be fulfilled for an *ideal phase state* [4] (whose definition requires $N \to \infty$, hence $\bar{n} = N/2 \to \infty$). It is worth mentioning again that this result is also in agreement with the proof given in [10], that the necessary condition to achieve the HL is $\Delta\hat{n}^2 \gtrsim \bar{n}^2$ which means that the states which give $\Delta\hat{n}^2 \ll \bar{n}^2$ cannot achieve the HL. In fact, in a phase state $\Delta\hat{n}^2 = \bar{n}^2/3 + \bar{n}/3$, hence $\Delta\hat{n}^2/\bar{n}^2 = (1 + 1/\bar{n})/3$ which fits the requirement introduced in [10].

Now, since a number state (which is the most sub-Poissonian) does not attain the HL while a phase state (being the most super-Poissonian) attains the HL, one would expect that its validity should depend on the nonclassical feature of the field state being measured. In addition, one naturally would expect that quantum aspects of a state, as this concerning with the HL, should depend on how much nonclassical the state is. However, as to be shown below, this is false. In fact, the number state is nonclassical and according to a theorem by Hillery [15], the phase state is also nonclassical. But, while the number state is the most nonclassical [16], what about the nonclassical depth of the phase state? Would it be so little nonclassical that it could behave as a classical one? The answer to this point comes from a calculation using a criterium introduced by Lee [16]. Accordingly, the R-function

$$R(z;\tau) \equiv \frac{1}{\tau}\int \frac{d^2w}{\pi}\exp\left(-\frac{1}{\tau}|z-w|^2\right)P(w,w^*) \tag{9}$$

measures how much nonclassical a field state is. $P(w,w^*)$ is the Glauber distribution function [17], given by [18]

$$P(w,w^*) = \int \frac{d^2u}{\pi}\exp\left[|w|^2 + |u|^2 + (wu^* - w^*u)\right] \times \langle -u|\hat{\rho}|u\rangle, \tag{10}$$

where $\hat{\rho}$ is the density operator representing the field state, τ is a continuous parameter interpolating the R-function from P-function to Q-function (Glauber–Sudarshan and Husimi functions, respectively). When $\tau = 0, 1/2$ and 1 the R-function coincides with P,

W and Q functions (Glauber–Sudarshan, Wigner and Husimi), respectively. Then, for quantum states the R-function interpolates from P to well behaved Q functions and τ varies in the interval $(0,1]$. Now, the minimum value τ_m in this interval, which makes the R-function well behaved was defined in [16] as a nonclassical depth of the field state represented by the operator $\hat{\rho}$. For the Fock states it was found that its nonclassical depth is maximum ($\tau_m = 1$), hence being the most quantum among all the nonclassical states. For the phase state, given in Eq. (6), the application of this criterium gives, after a few algebra using $\hat{\rho} = |\Theta_m\rangle\langle\Theta_m|$ in Eq. (4) of Ref. [19],

$$\begin{aligned} R_m(z;\tau) = {} & \frac{1}{(N+1)\tau} e^{-|z|^2/\tau} \\ & \times \left\{ \sum_{n=0}^{N} \left(\frac{\tau-1}{\tau}\right)^n L_n\left(\frac{|z|^2}{\tau(1-\tau)}\right) \right. \\ & + 2 \sum_{k=0}^{N-1} \sum_{n=k+1}^{N} \sqrt{\frac{k!}{n!}} \left(\frac{\tau-1}{\tau}\right)^k \left(\frac{|z|^2}{\tau}\right)^{n-k} \\ & \times L_k^{(n-k)}\left(\frac{|z|^2}{\tau(1-\tau)}\right) \\ & \left. \times \cos\bigl[(\theta_m - \zeta)(n-k)\bigr] \right\}, \qquad (11) \end{aligned}$$

where $L_n^m(z)$ stands for a Laguerre polynomial and $z = |z|e^{i\zeta}$. Numerical solution [8] shows that $\tau_m \to 1$ when the dimension N of the truncated Hilbert space is very large, as required for an ideal phase state [4]. So, as occurs for a number state the nonclassical depth of the phase state is also maximum ($\tau_m = 1$), showing that the validity of the HL is not connected with nonclassical depth of a state. At first sight, it seems somewhat surprising that some quantum aspect of a state (such as this HL) does not depend on its quantum depth. However, the foregoing results allow one to conclude that the crucial point behind the validity of the HL is essentially the statistical character of the field: *the HL requires that the field should be very super-Poissonian, irrespective of its degree of nonclassicality*.

Usually, experiments concerning with the nature of quantum objects reveal the mutual exclusiveness of particle–wave duality (PWD): they show what is called in literature as *sharp* PWD [20], the object exhibiting its *particle* or *wave* character, but not both. Traditional examples of these experiments occur in photodetection measurements (where light reveals its *particle* character) and the double-slit interference (light or particles revealing their *wave* character). However, it is also possible to observe simultaneously the particle and the wave behavior, i.e., the partial *which-path information* and the partial *interference*. This second possibility concerns with what is called *unsharp* PWD, as discussed in the literature [20,21].

In [9] the authors succeeded to relate formally (in a quantum nondemolition measurement setup) the particle property of the photon (which path the photon followed), represented by the signal-to-noise ratio (SNR) σ for the normalized quadrature phase measurement, to the photon wave property, represented by fringe visibility $v(\delta)$, which depends on the phase shift. For certain specific probe field states a simple relation, quantifying the complementarity principle, is available: for a coherent state, $v(\delta) = \exp[-\sigma^2/(4\sin^2(\delta/2))]$. However, there is no general expression like the Heisenberg or Schrödinger–Robertson uncertainty relations for noncommutative operators. For $\delta = \pi$ the optimal v versus σ is attained, $v = \exp[-\sigma^2/4]$, showing clearly the complementarity between wave-like ($v(\delta)$) and particle-like (σ) properties of the signal field. So, there is no a neat border between particle and wave behavior, the combination of the mutually exclusive classical concepts (which-path information and interference data) yielding the complete description of quantum objects. The discussed scheme is an example of incorporating both types of experiments (*sharp* and *unsharp* PWD). Further, this can be viewed as follows: for a probe field in a state $|\Psi(\xi)\rangle$ which interpolates between a number state $|N\rangle$ and a phase state $|\Theta_m\rangle$, $|\Psi(\xi)\rangle = \eta(\sqrt{1-\xi}\,|N\rangle + \sqrt{\xi}\,|\Theta_m\rangle)$, where η stands for normalization and $\xi \in [0,1]$ is an interpolating control parameter. When $\xi = 0$ ($\xi = 1$) we have that $|\Psi\rangle = |N\rangle$ ($|\Psi\rangle = |\Theta_m\rangle$), hence in these limits the state $|\Psi\rangle$ coincides either with $|N\rangle$ or $|\Theta_m\rangle$; $\langle N|\Theta_m\rangle \simeq 0$ for $N \gg 1$. Now, when $0 < \xi < 1$ the state $|\Psi\rangle$ is intermediate between these two complementary states. An alternative interpolating state with this characteristic was studied in [22]. Using the probe field in this interpolating state $|\Psi\rangle$ allows one to get an unified approach for the *sharp* and *unsharp* PWD. In fact, while the cases $\xi = 0$ (visibility (5) of state $|N\rangle$ is δ-independent) and $\xi = 1$ (state $|\Theta_m\rangle$ incorporates the phase δ and the visibility is quite sensible to it) correspond to the mutually exclusive (*sharp*) scenery, the case $0 < \xi < 1$ corresponds to the inclusive (*unsharp*) scenery since now the phase of the state $|\Psi\rangle$ is neither completely random nor completely well-defined: its determination in the experiment being partial, leading to partial *which-path information* and partial *interference*. Here, it is worth mentioning that the first experiment accessing both the particle and wave nature of light was realized recently by Foster et al. [23].

A final comment is pertinent: considering a phase state may sound not very convincing since an ideal phase state is unphysical because it has infinite mean energy; hence more realistically one resorts to a truncated phase state. However, this characteristic is not exclusive of the ideal phase state: it should be remembered that the precise measurement of the position of a particle would give it an infinite variance in momentum, which is also unphysical, but the position of a particle is obviously a measurable observable quantity in quantum mechanics with a nonzero variance, that can be described by the "truncated" positional state, i.e., $|x\rangle = \int_{-p_0}^{p_0} e^{ipx}|p\rangle\, dp/\sqrt{2\pi}$.

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