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# Nonstationary Casimir effect in cavities with two resonantly coupled modes

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## Abstract

We study the peculiarities of the nonstationary Casimir effect (creation of photons in cavities with moving boundaries) in the special case of two resonantly coupled modes with frequencies  $\omega_0$  and  $(3 + \Delta)\omega_0$ , parametrically excited due to small amplitude oscillations of the ideal cavity wall at the frequency  $2\omega_0(1 + \delta)$  (with  $|\delta|, |\Delta| \ll 1$ ). The effects of thermally induced oscillations in time dependences of the mean numbers of created photons and the exchange of quantum purities between the modes are discovered. Squeezing and photon distributions in each modes are calculated for initial vacuum and thermal states. A possibility of compensation of detunings is shown. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Classical and quantum phenomena in cavities with moving boundaries attracted attention of many researchers for a long time (see review [1]). Especially popular this topic became in the last decade, being known now under the names *nonstationary Casimir effect* [2], *dynamical Casimir effect* [3], or *mirror (motion) induced radiation* [4,5]. One of several theoretical results obtained in the last years was the prediction of the exponential growth of the energy of the field under the resonance conditions, when the wall performs

vibrations at the frequency which is a multiple of the unperturbed field eigenfrequency [5–8].

In most papers, the special case of the one-dimensional cavity with an equidistant spectrum of unperturbed field modes was studied. Then an infinite number of modes are excited due to an intermode interaction (resulting from the Doppler effect on the moving boundary), and the total number of created photons depends on time as  $t^2$  (but the total energy increases exponentially) [1,8–11]. A more realistic case of a three-dimensional cavity was considered, e.g., in [12–14]. In particular, simple analytical solutions describing the resonance creation of photons in cavities with totally nonequidistant spectra, when only one mode is in resonance with the vibrating wall, were given in [1,7,8]. However, the case of a few resonantly interacting modes is also of a great interest. In particular, the interaction between the excited field mode and the de-

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tector, approximated either by a harmonic oscillator or by a two-level atom, was considered in [7,8,15–19].

It was shown recently [20], that for rectangular three- and two-dimensional cavities, there exist special configurations, when two (or more) modes can occur in resonance with the moving wall and between themselves. For example, the eigenmode spectrum in a cubical cavity of length  $L$  is given by the formula  $\omega_{klm} = (\pi c/L)\sqrt{k^2 + l^2 + m^2}$ . Since  $\omega_{511} = 3\omega_{111}$ , both the modes, {111} and {511}, will be excited if the wall oscillates at the frequency  $2\omega_{111}$ . In such a case, it appears [20] that the energies of both the modes increase in time exponentially (with some additional oscillations), but the increment of the exponential growth is twice smaller than in the case of a single resonance mode.

Our goal is to give analytical solutions describing the case of two resonantly interacting modes in a more or less generic situation, without specializing the form of the cavity. We take into account the possibility of detuning between the frequency of the vibrating wall and the twice fundamental field eigenfrequency, and detuning between the frequency of the second mode and the triple fundamental eigenfrequency (however, we do not consider the effects of damping). Using the method of slowly varying amplitudes, we obtain explicit solutions of equations of motion for the canonical operators describing the fields in coupled modes, which enable us to calculate the mean energy and the invariant uncertainty product for each mode. We analyze the evolution of the photon distribution functions for each mode and show the possibility of strong *squeezing* in the long-time limit (for other configurations this was done in [7,8,10]).

## 2. General solution for two coupled modes

We use the Hamiltonian approach proposed by Law [6] and developed in [21] (for other references see [1]). Consider the scalar massless field  $\Phi(\mathbf{r}, t)$ , satisfying the wave equation  $\Phi_{tt} = \nabla^2 \Phi$  inside the cavity and the Dirichlet boundary condition  $\Phi = 0$  on the boundary. We assume that we know the complete orthonormalized set of eigenfunctions (and eigenfrequencies) of the Laplace equation  $\nabla^2 f_\alpha(\mathbf{r}) + \omega_\alpha^2 f_\alpha(\mathbf{r}) = 0$  in the case of stationary cavity. Now suppose that a part of the boundary is a plane surface moving according to

a *prescribed* law of motion  $L(t)$  (for the most recent study of the case when  $L(t)$  is a *dynamical variable* due to the back reaction of the field see [22]). Expanding the field  $\Phi(\mathbf{r}, t)$  over “instantaneous” eigenfunctions  $f_\alpha(\mathbf{r}; L(t))$ ,

$$\Phi(\mathbf{r}, t) = \sum_{\alpha} q_{\alpha}(t) f_{\alpha}(\mathbf{r}; L(t)), \quad (1)$$

we satisfy automatically the boundary conditions. Then the dynamics of the field is described completely by the dynamics of the generalized coordinates  $q_{\alpha}(t)$ , which, in turn, can be derived from the *time-dependent Hamiltonian* [21]

$$H(t) = \frac{1}{2} \sum_{\alpha} [p_{\alpha}^2 + \omega_{\alpha}^2(L(t)) q_{\alpha}^2] + \frac{\dot{L}(t)}{L(t)} \sum_{\alpha \neq \beta} p_{\alpha} m_{\alpha\beta} q_{\beta} \quad (2)$$

with antisymmetrical time-independent coefficients

$$m_{\alpha\beta} = -m_{\beta\alpha} = L \int dV \frac{\partial f_{\alpha}(\mathbf{r}; L)}{\partial L} f_{\beta}(\mathbf{r}; L). \quad (3)$$

For example, in the case of a rectangular three-dimensional cavity, the eigenmodes are the well known products of three sine functions like  $\sin(\pi k_x x/L_x)$ , labeled by three natural numbers  $k_x, k_y, k_z$ . If one surface of the parallelepiped, perpendicular to the  $x$ -axis, moves in the  $x$ -direction (so that the  $L_x$ -dimension of the cavity is a function of time), then [20]

$$m_{\mathbf{k}\mathbf{j}} = (-1)^{k_x + j_x} \frac{2k_x j_x}{j_x^2 - k_x^2} \delta_{k_y j_y} \delta_{k_z j_z}. \quad (4)$$

We are interested in the case when one of the cavity walls performs small oscillations with the frequency  $\Omega$  close to the double frequency of some unperturbed mode  $\omega_1^{(0)} \equiv 1$  (i.e., we normalize all frequencies by  $\omega_1^{(0)}$ ), so that the time-dependent frequency  $\omega_1(t)$  reads

$$\omega_1(t) = 1 + 2\epsilon \cos(2\bar{\omega}t), \quad \bar{\omega} = 1 + \delta, \quad (5)$$

where we assume that  $|\delta| \ll 1$  and  $|\epsilon| \ll 1$ . Also we suppose that the unperturbed field frequency spectrum includes the frequency  $\omega_3^{(0)} = 3 + \Delta$  with  $|\Delta| \ll 1$ , but it does not contain frequencies close to  $5\omega_1^{(0)}$ . Then we have the case of two resonantly interacting modes, and it is sufficient to consider only the part of the total

Hamiltonian (2) related to these modes:

$$H_{13} = \frac{1}{2}(p_1^2 + p_3^2) + \frac{1}{2}[1 + 4\epsilon \cos(2\bar{\omega}t)]x_1^2 + \frac{1}{2}[9 + 6\Delta + \tilde{\epsilon} \cos(2\bar{\omega}t)]x_3^2 + 3\mu\epsilon \sin(2\bar{\omega}t)(p_1x_3 - p_3x_1). \quad (6)$$

The constant parameter  $\mu$  is proportional to the coefficient  $m_{12}$  in (2). Writing (6) we have neglected the second-order terms with respect to  $\epsilon$  and  $\Delta$ . Since the time-dependent frequency shift  $\omega_1(t) - \omega_1^{(0)} \sim L(t) - L_0$  is chosen to be proportional to  $\cos(2\bar{\omega}t)$ , the last term in (6), which is proportional to  $\dot{L}(t)$ , must depend on time as  $\sin(2\bar{\omega}t)$ . All numerical coefficients in (6) are chosen in order to avoid an appearance of fractions in formulas below. Parameter  $\tilde{\epsilon}$  has the same order of magnitude as  $\epsilon$ , but it does not affect the solution in the zeroth-order approximation (in which we are interested here), as will be shown below.

Hamiltonian (6) results in the following differential equations for the generalized coordinates  $x_1$  and  $x_3$  (we neglect corrections of the second order):

$$\ddot{x}_1 = -[1 + 4\epsilon \cos(2\bar{\omega}t)]x_1 + 24\mu\epsilon [\cos(2\bar{\omega}t)x_3 + \sin(2\bar{\omega}t)\dot{x}_3], \quad (7)$$

$$\ddot{x}_3 = -[9 + 6\Delta + \tilde{\epsilon} \cos(2\bar{\omega}t)]x_3 - 24\mu\epsilon [\cos(2\bar{\omega}t)x_1 + \sin(2\bar{\omega}t)\dot{x}_1]. \quad (8)$$

We solve Eqs. (7) and (8), using the method of slowly varying amplitudes [23–25] (which was applied earlier in studies [7–9,26]). Namely, we look for the solutions in the form

$$x_k(t) = \xi_k^+(t)e^{ik\bar{\omega}t} + \xi_k^-(t)e^{-ik\bar{\omega}t}, \quad k = 1, 3, \quad (9)$$

where each coefficient  $\xi_k^\pm$  is a *slowly varying function of time*, whose derivatives are proportional to the small parameters  $\epsilon, \delta, \Delta$ , so that we can neglect the second-order derivatives  $\ddot{\xi}_k^\pm$ . Then, equating the coefficients at  $\exp[\pm i\bar{\omega}t]$  in Eq. (7) and coefficients at  $\exp[\pm 3i\bar{\omega}t]$  in Eq. (8) (and neglecting the terms proportional to squares and products of small coefficients  $\epsilon, \delta, \Delta$ ), we obtain a set of equations with *constant* coefficients for the slowly varying amplitudes. It is convenient to write this set in the matrix form  $d\mathbf{v}/dt = \mathcal{A}\mathbf{v}$ , introducing

vector  $\mathbf{v} = (\xi_1^+, \xi_1^-, \xi_3^+, \xi_3^-)$  and matrix

$$\mathcal{A} = \begin{pmatrix} -i\delta & i\epsilon & 12i\mu\epsilon & 0 \\ -i\epsilon & i\delta & 0 & -12i\mu\epsilon \\ 4i\mu\epsilon & 0 & i(\Delta - 3\delta) & 0 \\ 0 & -4i\mu\epsilon & 0 & -i(\Delta - 3\delta) \end{pmatrix}. \quad (10)$$

Strictly speaking, function  $x_1(t)$  contains also terms proportional to  $\exp[\pm 3i\bar{\omega}t]$ , as well as function  $x_3(t)$  contains terms proportional to  $\exp[\pm i\bar{\omega}t]$ . However, the amplitudes of these additional terms are proportional to the small parameters  $\epsilon, \delta, \Delta$ , so we neglect them, confining ourselves to the principal terms whose amplitudes are of the zeroth order with respect to small parameters. Just for this reason, the parameter  $\tilde{\epsilon}$  does not appear in matrix  $\mathcal{A}$ : while the term  $4\epsilon \cos(2\bar{\omega}t)$  standing at  $x_1$  in Eq. (7) connects the slowly varying amplitudes  $\xi_1^+$  and  $\xi_1^-$ , a similar term  $\tilde{\epsilon} \cos(2\bar{\omega}t)$  standing at  $x_3$  in (8) does not connect the amplitudes  $\xi_3^+$  and  $\xi_3^-$  between themselves, but it gives small corrections to the amplitudes of higher-order terms in  $x_3$ , oscillating with frequencies  $\bar{\omega}$  and  $5\bar{\omega}$ . A more strict (from the mathematical point of view) method, based on the multiple scale analysis, was exposed in [20]. In the zeroth-order approximation it leads to the same results.

Matrix  $\mathcal{A}$  has four eigenvalues,  $\pm\lambda_\pm$ , where

$$\lambda_\pm = \frac{1}{\sqrt{2}}\sqrt{a \pm \sqrt{c}} \quad (11)$$

$$\equiv \frac{1}{2}\left(\sqrt{a + \sqrt{b}} \pm \sqrt{a - \sqrt{b}}\right), \quad (12)$$

$$a = \epsilon^2(1 - \nu) - \delta^2 - (\Delta - 3\delta)^2, \quad (13)$$

$$b = [2(\delta - \epsilon)(\Delta - 3\delta) + \nu\epsilon^2] \times [2(\delta + \epsilon)(\Delta - 3\delta) + \nu\epsilon^2], \quad (14)$$

$$c \equiv a^2 - b = 2\epsilon^2\nu[(\Delta - 4\delta)^2 - \epsilon^2] + [\epsilon^2 + (\Delta - 2\delta)(\Delta - 4\delta)]^2, \quad (15)$$

$$\nu \equiv 96\mu^2. \quad (16)$$

Comparing our matrix  $\mathcal{A}$  (10) with a similar matrix found in [20] for the rectangular cavity, one can verify that if modes  $\{k_x, m, n\}$  and  $\{j_x, m, n\}$  are in resonance, then  $\mu = j_x/(12k_x)$  (different phases of elements of matrices in [20] and here are due to different choices of trigonometrical dependences in function  $L(t)$ : we use  $\cos$ -function instead of  $\sin$ -function in [20]). In particular, for the modes  $\{111\}$  and  $\{511\}$

of the cubical cavity we have  $\nu = 50/3$ . Due to this explicit example (and other examples related to rectangular cavities), we assume hereafter that parameter  $\nu$  is large, so that it satisfies the inequality  $\nu \gg 1$ .

### 3. Exact resonance

We start with the simplest case of the exact resonance,  $\delta = \Delta = 0$ . Then formula (12) yields

$$\lambda_{\pm}^{(\nu)} = \frac{\epsilon}{2}(1 \pm \sqrt{1 - 2\nu}). \quad (17)$$

If  $\nu = 0$  (no intermode coupling), then we obtain the increment of the exponential growth  $\lambda_{+}^{(0)} = \epsilon$ , in accordance with the solutions of the single-mode problem found in [7,8,26]. On the contrary, for  $\nu > 1/2$  the increment (real part of  $\lambda_{\pm}$ ) does not depend on the form of the cavity (which is “hidden” in the value of  $\nu$ ), being exactly twice smaller than in the single-mode case. In the special case of rectangular cavities this result was obtained in [20].

After some straightforward calculations we arrive at the following explicit expressions for the time dependences of the generalized coordinates (as far as we neglect all corrections of the order of  $\epsilon$  in the amplitude coefficients, we may identify the canonical momenta with velocities,  $p_k = \dot{x}_k$ ,  $k = 1, 2$ ):

$$\begin{aligned} x_1(t) = & x_1(0) \left[ C_1^- \cos(\rho\tau) + S_1^- \frac{\sin(\rho\tau)}{\rho} \right] \\ & - p_1(0) \left[ S_1^- \cos(\rho\tau) + C_1^- \frac{\sin(\rho\tau)}{\rho} \right] \\ & + 8\mu \frac{\sin(\rho\tau)}{\rho} [3S_1^- x_3(0) + C_1^- p_3(0)], \end{aligned} \quad (18)$$

$$\begin{aligned} x_3(t) = & x_3(0) \left[ C_3^+ \cos(\rho\tau) - S_3^+ \frac{\sin(\rho\tau)}{\rho} \right] \\ & + \frac{1}{3} p_3(0) \left[ S_3^+ \cos(\rho\tau) - C_3^+ \frac{\sin(\rho\tau)}{\rho} \right] \\ & - 8\mu \frac{\sin(\rho\tau)}{\rho} [S_3^+ x_1(0) - C_3^+ p_1(0)], \end{aligned} \quad (19)$$

where

$$\begin{aligned} \tau &\equiv \frac{1}{2}\epsilon t, & \rho &= \sqrt{2\nu - 1}, \\ C_k^{\pm}(\tau; t) &= \cosh \tau \cos(k\bar{\omega}t) \pm \sinh \tau \sin(k\bar{\omega}t), \\ S_k^{\pm}(\tau; t) &= \sinh \tau \cos(k\bar{\omega}t) \pm \cosh \tau \sin(k\bar{\omega}t). \end{aligned} \quad (20)$$

Similar expressions for the time dependent canonical momenta can be easily obtained by differentiating Eqs. (18), (19) with respect to the “fast time”  $t$ , considering the “slow time” variable  $\tau$  as an independent parameter and neglecting corrections of the order of  $\epsilon, \delta, \Delta$  in the amplitude coefficients. The following relations hold in this approximation:

$$\frac{\partial C_k^{\pm}}{\partial t} = \pm k S_k^{\mp}, \quad \frac{\partial S_k^{\pm}}{\partial t} = \pm k C_k^{\mp}.$$

Symbols  $x_k, p_k$  in Eqs. (18), (19) can be considered both as classical variables and quantum operators in the Heisenberg picture, due to the linearity of the problem (or due to the quadratic nature of Hamiltonian (2)). Using Eqs. (18), (19) and their momentum counterparts, one can calculate mean values of squares and products of canonical variables (operators) at any moment of time, provided such mean values were known at the initial moment  $t = 0$ . We confine ourselves to the simplest case, when initially the field modes were in thermal states with the mean photon numbers  $(\theta_1 - 1)/2$  and  $(\theta_3 - 1)/2$ , where  $\theta_k = \coth(k\beta/2)$ ,  $\beta$  being inverse absolute temperature in dimensionless units. One can check the relations

$$\theta_{31} \equiv \frac{\theta_3}{\theta_1} = \theta_{13}^{-1} = \frac{\theta_1^2 + 3}{3\theta_1^2 + 1}, \quad 1 \geq \theta_{31} \geq \frac{1}{3}.$$

The mean energies in each mode,  $\mathcal{E}_k = (1/2)\langle p_k^2 + \omega_k^2 x_k^2 \rangle$ , depend on time as follows:

$$\begin{aligned} \mathcal{E}_1 = \frac{\theta_1}{2} \left\{ \cosh(2\tau) \left[ \frac{\sin^2(\rho\tau)}{\rho^2} (1 + 2\nu\theta_{31}) \right. \right. \\ \left. \left. + \cos^2(\rho\tau) \right] + \sinh(2\tau) \frac{\sin(2\rho\tau)}{\rho} \right\}, \end{aligned} \quad (21)$$

$$\begin{aligned} \mathcal{E}_3 = \frac{3\theta_3}{2} \left\{ \cosh(2\tau) \left[ \frac{\sin^2(\rho\tau)}{\rho^2} (1 + 2\nu\theta_{13}) \right. \right. \\ \left. \left. + \cos^2(\rho\tau) \right] - \sinh(2\tau) \frac{\sin(2\rho\tau)}{\rho} \right\}. \end{aligned} \quad (22)$$

For rectangular cavities (and, perhaps, for others),  $\nu \gg 1$  in the case of intermode resonance. Then  $\rho^2 \approx 2\nu \gg 1$ , so that Eqs. (21) and (22) can be simplified:

$$\mathcal{E}_1 \approx \frac{1}{2} \cosh(2\tau) [\theta_1 \cos^2(\rho\tau) + \theta_3 \sin^2(\rho\tau)],$$

$$\mathcal{E}_3 \approx \frac{3}{2} \cosh(2\tau) [\theta_3 \cos^2(\rho\tau) + \theta_1 \sin^2(\rho\tau)].$$

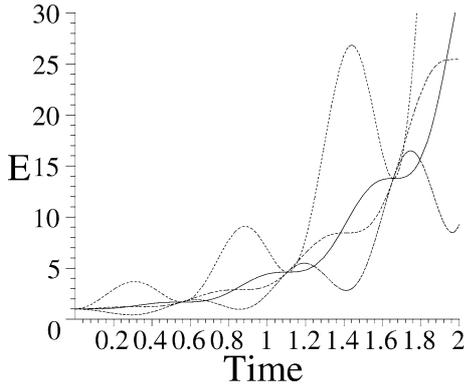


Fig. 1. Mean energies (normalized by their initial values at  $\tau = 0$ ) of the first and the third modes of the cubic cavity with  $\nu = 50/3$  in the case of the strict resonance, versus the dimensionless “slow time”  $\tau$  (20). The order of the curves in the interval of time between 1.4 and 1.6 is as follows (from bottom to top): the first mode for  $\theta_{31} = 1/3$  (high-temperature initial states); the first mode for  $\theta_1 = \theta_3 = 1$  (vacuum initial state); the third mode for  $\theta_1 = \theta_3 = 1$ ; the third mode for  $\theta_{13} = 3$ .

For the initial vacuum states ( $\theta_1 = \theta_3 = 1$ ) we have a monotonous growth of energy and number of photons in each mode:  $\mathcal{E}_k \approx (\omega_k^{(0)}/2) \cosh(2\tau)$ ,  $k = 1, 3$ . On the contrary, for high-temperature initial states (with equal temperatures),  $\theta_1 = 3\theta_3 \gg 1$ , which results in strong oscillations of mean energies and numbers of photons:

$$\mathcal{E}_1 \approx \frac{\theta_1}{2} \cosh(2\tau) \left[ 1 - \frac{2}{3} \sin^2(\rho\tau) \right],$$

$$\mathcal{E}_3 \approx \frac{\theta_1}{2} \cosh(2\tau) [1 + 2 \sin^2(\rho\tau)].$$

In Fig. 1 we show normalized time dependences of the mean energies (calculated with the aid of complete formulas (21) and (22)) in each resonant mode in the low- and high-temperature limits. Note that one has  $\theta_1 \approx 140$ , if  $L_0 = 1$  cm and  $T = 300$  K.

Since Hamiltonian (2) is *quadratic* with respect to canonical coordinates and momenta, the initial thermal state is transformed with time to a generic *Gaussian quantum state* (whose density matrix or Wigner function is a Gaussian exponential). The single-mode density matrices of such states are completely characterized (in the case of zero mean values of quadrature components), besides the mean energy, by the *invariant uncertainty product* (IUP)

$$\mathcal{D} \equiv \langle x^2 \rangle \langle p^2 \rangle - \langle (xp + px)/2 \rangle^2. \quad (23)$$

For  $\nu > 1/2$  this quantity varies in time periodically. For the first mode,

$$\mathcal{D}_1 = \frac{\theta_1^2}{4} \left[ \cos^4(\rho\tau) + \sin^2(2\rho\tau) \frac{2\nu\theta_{31} - 1}{2(2\nu - 1)} + \sin^4(\rho\tau) \left( \frac{2\nu\theta_{31} + 1}{2\nu - 1} \right)^2 \right]. \quad (24)$$

For another excited mode one should interchange indices 1 and 3 in (24).

The *purity* of quantum Gaussian states is expressed through IUP as  $\text{Tr}(\hat{\rho}^2) = (4\mathcal{D})^{-1/2}$  (here  $\hat{\rho}$  is the statistical operator of the state). For initial vacuum states,

$$\mathcal{D}_{1,3} = \frac{1}{4} \left[ 1 + \frac{8\nu}{(2\nu - 1)^2} \sin^4(\rho\tau) \right], \quad (25)$$

so that the states remain practically pure, if  $\nu \gg 1$ . This is a significant difference from the case of a single resonance mode coupled to a harmonic oscillator detector, studied in [7,8]. If  $\theta_1 \neq \theta_3$  (nonzero initial temperature) and  $\nu \gg 1$ , then

$$\mathcal{D}_1 \approx \frac{1}{4} [\theta_1 \cos^2(\rho\tau) + \theta_3 \sin^2(\rho\tau)]^2,$$

so that coupled modes *exchange their purities* at the moments when  $\sin(\rho\tau) = 1$ .

The *squeezing coefficient*, defined as the ratio of the minimal value of any canonical variance ( $\sigma_x$  or  $\sigma_p$ ) for the period of fast oscillations (with frequencies  $\bar{\omega}$  or  $3\bar{\omega}$ ) to the vacuum value ( $(2\omega_k^{(0)})^{-1}$  or  $\omega_k^{(0)}/2$ ), can be expressed through  $\tilde{\mathcal{E}} \equiv \mathcal{E}/\omega_k^{(0)}$  and  $\mathcal{D}$  as [27,28]

$$s = \frac{2\mathcal{D}}{\tilde{\mathcal{E}} + \sqrt{\tilde{\mathcal{E}}^2 - \mathcal{D}}}. \quad (26)$$

For thermal states,  $s(0) = \theta$ , but asymptotically one can obtain any desired degree of squeezing (even for initial high-temperature states) in each resonant mode, because  $s \approx \mathcal{D}(\tau)/\tilde{\mathcal{E}}(\tau)$  for  $\tau \gg 1$ . Evidently,  $\tilde{\mathcal{E}}$  is nothing but the mean number of photons in the mode, if  $\tilde{\mathcal{E}} \gg 1$ . The variance of the photon distribution in the Gaussian states is given by the formula [29]

$$\sigma_n = 2\tilde{\mathcal{E}}^2 - \mathcal{D} - 1/4.$$

The photon statistics is strongly super-Poissonian for  $\tau \gg 1$ , when  $\sigma_n/\langle n \rangle \approx 2\tilde{\mathcal{E}} \gg 1$ . Nonetheless, the quantum states of each excited mode become highly nonclassical for  $\tau \gg 1$ . It is seen from the photon

distribution function (PDF), which can be expressed in terms of two parameters,  $\tilde{\mathcal{E}}$  and  $\mathcal{D}$ , for any Gaussian state with zero mean values of quadrature variables [29]:

$$\mathcal{P}_n = \frac{2}{\sqrt{1+4\tilde{\mathcal{E}}+4\mathcal{D}}} \left( \frac{1+4\mathcal{D}-4\tilde{\mathcal{E}}}{1+4\mathcal{D}+4\tilde{\mathcal{E}}} \right)^{n/2} \times P_n \left( \frac{4\mathcal{D}-1}{\sqrt{(4\mathcal{D}+1)^2-16\tilde{\mathcal{E}}^2}} \right). \quad (27)$$

Here  $P_n(z)$  is the Legendre polynomial. Initially  $\tilde{\mathcal{E}}^2(0) = \mathcal{D}(0)$ , and (27) is a monotonous geometric distribution,  $\mathcal{P}_n(0) = 2(\theta-1)^n/(\theta+1)^{n+1}$ . Since we are interested in the cases of large numbers of photons created due to the parametric resonance, it is convenient to use asymptotical forms of the exact distribution (27) for  $n \gg 1$ . Note that the argument of the Legendre polynomial in (27) is always outside the interval  $(-1, 1)$ , being equal to 1 only at the initial moment (for thermal states). With the course of time this argument increases to  $\infty$ , becomes pure imaginary, and asymptotically goes to zero, when  $\tilde{\mathcal{E}} \gg \mathcal{D}$ . Therefore it is convenient to use the asymptotical formula for  $n \gg 1$  [30],

$$\mathcal{P}_n(\cosh \xi) \approx \left( \frac{\xi}{\sinh \xi} \right)^{1/2} I_0([n+1/2]\xi) \quad (28)$$

(where  $I_0(z)$  is the modified Bessel function), because it holds even for complex values of variable  $\xi$ , provided  $\text{Re } \xi \geq 0$  and  $|\text{Im } \xi| < \pi$ . We are interested in the case when the mean energy of each mode significantly exceeds its initial value, i.e.,  $\tilde{\mathcal{E}} \gg \sqrt{\mathcal{D}}$ . If  $\mathcal{D} \gg 1$  (high-temperature initial states), then it is possible that  $\tilde{\mathcal{E}} < \mathcal{D} + 1/4$ . In such a case,  $\xi$  is real and large, so that one can replace the modified Bessel function  $I_0(x)$  by its asymptotical expression for  $x \gg 1$ ,  $I_0(x) \approx (2\pi x)^{-1/2} \exp(x)$ . Moreover, one can use either of approximate equalities  $\exp(x) \approx 2 \cosh(x) \approx 2 \sinh(x)$ .

When  $\tilde{\mathcal{E}} > \mathcal{D} + 1/4$  (this is just the regime when *squeezing* happens), the argument of the Legendre polynomial becomes pure imaginary, and the variable  $\xi$  becomes complex:  $\xi = -i\pi/2 + y$ , with  $y > 0$ . Then one should replace the modified Bessel function in (28) by the usual Bessel function  $J_0([n+1/2][\pi/2 + iy])$ . Since the absolute value of the argument of this function is also large, one can replace  $J_0(x)$  by  $(\pi x/2)^{-1/2} \cos(x - \pi/4)$ . In this case, we

have different expressions for even and odd values of index  $n$  (this is especially clear from the initial formula (27), when the argument of the Legendre polynomial is close to zero for  $\tilde{\mathcal{E}} \gg \mathcal{D}$ ). Finally, we arrive at the following asymptotical expression, which holds both for real and imaginary values of the argument of the Legendre polynomial in (27) (provided  $\tilde{\mathcal{E}}^2 \gg \mathcal{D}$ ):

$$\mathcal{P}_n \approx \frac{\sqrt{2}|1+4\mathcal{D}-4\tilde{\mathcal{E}}|^{n/2}}{\sqrt{\pi n \tilde{\mathcal{E}}(1+4\mathcal{D}+4\tilde{\mathcal{E}})^{(n+1)/2}}} \times \begin{cases} \cosh([n+1/2] \ln |\chi|), & n \text{ even,} \\ \sinh([n+1/2] \ln |\chi|), & n \text{ odd,} \end{cases} \quad (29)$$

where

$$\chi = \frac{4\mathcal{D}-1+4\sqrt{\tilde{\mathcal{E}}^2-\mathcal{D}}}{\sqrt{(4\mathcal{D}+1)^2-16\tilde{\mathcal{E}}^2}}. \quad (30)$$

If the ratio  $\tilde{\mathcal{E}}/\mathcal{D}$  is not very large, then  $\ln |\chi|$  is not small, and one can replace  $\cosh$  and  $\sinh$  functions in (29) by exponentials. In this case we have a nonoscillating “quasigeometric” distribution, i.e., a geometric distribution modified by a slowly decreasing factor  $n^{-1/2}$ :

$$\mathcal{P}_n \approx \frac{1}{\sqrt{2\pi n \tilde{\mathcal{E}}}} \left( \frac{4\mathcal{D}-1+4\sqrt{\tilde{\mathcal{E}}^2-\mathcal{D}}}{4\mathcal{D}+1+4\tilde{\mathcal{E}}} \right)^{n+1/2}. \quad (31)$$

On the contrary, if  $\tilde{\mathcal{E}} \gg \mathcal{D}$  (then  $\chi$  is pure imaginary and small), we have a strongly oscillating distribution for  $n < \tilde{\mathcal{E}}/\mathcal{D}$  (typical for squeezed states):

$$\mathcal{P}_n \approx \sqrt{\frac{2}{\pi n \tilde{\mathcal{E}}}} \left( 1 - \frac{4\mathcal{D}+1}{2\tilde{\mathcal{E}}} \right)^{n/2} \times \begin{cases} \cosh\left(n \frac{4\mathcal{D}-1}{4\tilde{\mathcal{E}}}\right), & n \text{ even,} \\ \sinh\left(n \frac{4\mathcal{D}-1}{4\tilde{\mathcal{E}}}\right), & n \text{ odd.} \end{cases} \quad (32)$$

Only the “tail” of distribution (32) does not oscillate (and does not depend on  $\mathcal{D}$ ):

$$\mathcal{P}_n \approx \frac{\exp[-n/(2\tilde{\mathcal{E}})]}{\sqrt{2\pi n \tilde{\mathcal{E}}}}, \quad n \frac{4\mathcal{D}-1}{4\tilde{\mathcal{E}}} \gg 1. \quad (33)$$

For the vacuum initial state, the PDF oscillates from the beginning. For  $\tau \gg 1$  and  $\nu \gg 1$  we can write (for each excited mode)

$$\mathcal{P}_n \approx e^{-\tau} \sqrt{\frac{8}{\pi n}} (1 - 4e^{-2\tau})^{n/2}$$

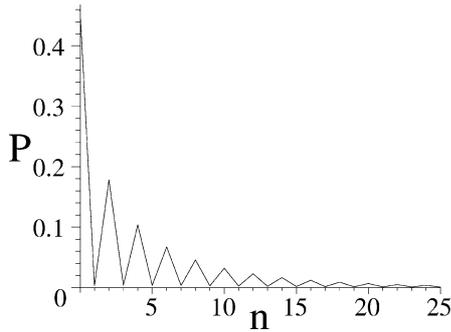


Fig. 2. The photon distribution function  $\mathcal{P}(n)$  for the first mode and the initial vacuum state, under the condition of the strict resonance, for the “slow time”  $\tau = 5\pi/(2\rho)$  and  $\nu = 50/3$ .

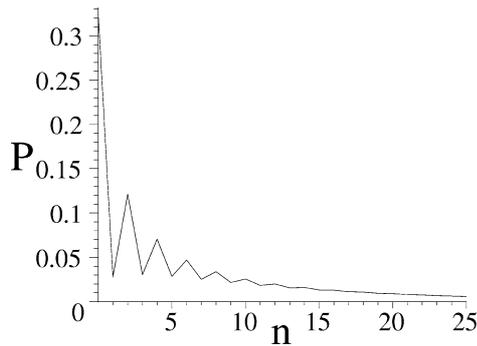


Fig. 3. The photon distribution function  $\mathcal{P}(n)$  for the first mode and the initial thermal state with  $\theta_1 = 5$  (i.e., for  $\langle n(0) \rangle = 2$  in the first mode), under the condition of the strict resonance, for the “slow time”  $\tau = 5\pi/(2\rho)$  and  $\nu = 50/3$ .

$$\times \begin{cases} \cosh\left(\frac{2n}{\nu} \sin^4(\rho\tau)e^{-2\tau}\right), & n \text{ even,} \\ \sinh\left(\frac{2n}{\nu} \sin^4(\rho\tau)e^{-2\tau}\right), & n \text{ odd.} \end{cases}$$

Oscillations exist for any  $n$  at the moments of time when  $\sin(\rho\tau) = 0$ . Two examples of the photon distribution functions for the initial vacuum and thermal states (calculated using the exact formula (27)) are given in Figs. 2 and 3.

#### 4. Influence of detunings

Now let us consider the generic case of nonzero detunings. Hereafter we use the normalized parameters  $\tilde{\delta} \equiv \delta/\epsilon$  and  $\tilde{\Delta} \equiv \Delta/\epsilon$ . The condition of the parametric resonance is that at least one of the eigenvalues  $\lambda_{\pm}$ , given by Eqs. (11) or (12), has nonzero real part. Since

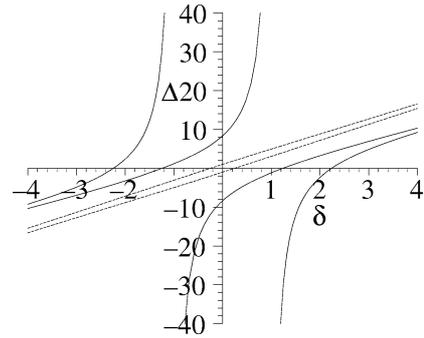


Fig. 4. Regions of photon generation in the plane  $(\tilde{\delta}, \tilde{\Delta})$  of the detuning parameters (tildes have been omitted). The region of “symmetrical generation”  $|\tilde{\Delta} - 4\tilde{\delta}| < \eta_c$  is between two dashed (almost straight) lines. Two regions of “asymmetrical generation” are confined with hyperbolas drawn as continuous curves.

we assume that  $\nu > 1$ , the coefficient  $a$  (13) is always negative. Looking at the eigenvalues in form (11), we see that there are two possibilities.

The first one is to choose  $c < 0$ . Then complex number  $a + \sqrt{c}$  has nonzero imaginary part, so its square root inevitably has nonzero real part. Designating  $\tilde{\Delta} - 4\tilde{\delta} = \eta$ , we see that  $c < 0$  for  $|\eta| < \eta_c$ , where the critical value  $\eta_c$  (which depends on  $\nu$  and  $\tilde{\delta}$ ) must be obviously less than 1. If  $\nu \gg 1$  and  $|\delta| \sim 1$ , then  $\eta_c$  is close to 1, with corrections of the order of  $\nu^{-1}$ . If  $|\tilde{\delta}| \gg 1$ , then for  $1 \sim |\eta| \ll |\tilde{\delta}|$  one can write  $c \approx 2\nu(\eta^2 - 1) + 4\tilde{\delta}^2\eta^2$  (taking into account that  $\tilde{\Delta} - 2\tilde{\delta} = 2\tilde{\delta} + \eta \approx 2\tilde{\delta}$ ). In this way we obtain the following approximate inequality, describing the region of resonance in the parameter plane  $\tilde{\delta}, \tilde{\Delta}$  (see Fig. 4):

$$|\tilde{\Delta} - 4\tilde{\delta}| < \eta_c \approx \sqrt{\frac{\nu}{\nu + 2\tilde{\delta}^2}}. \quad (34)$$

Consequently, one can always excite both the modes, compensating one detuning by another. For example, if  $\tilde{\Delta} = 4\tilde{\delta}$  and  $\nu \gg 1$ , then  $b = a^2 - c$  with  $|c| \ll a^2$ , so that  $\sqrt{b} \approx |a| - c/(2|a|)$ . In this case Eq. (12) yields

$$\lambda_{\pm} \approx \frac{\epsilon}{2} \left[ \sqrt{\frac{\nu}{\nu + 2\tilde{\delta}^2}} \pm i\sqrt{2(\nu + 2\tilde{\delta}^2)} \right]. \quad (35)$$

As far as  $\tilde{\delta}^2 \ll \nu$ , the maximal value of the increment coefficient  $\text{Re}(\lambda_{\pm})$  is practically the same as in the case of strict resonance (17). Energies of both modes have the same order of magnitude (as in the strict resonance case), therefore this regime of excitation can be named “symmetrical”. However, with increase of

$|\tilde{\delta}|$  the increment of energy growth decreases (by the same law as the resonance width  $\eta_c$ ), and for  $\tilde{\delta}^2 \gg \nu$  we have  $\text{Re}(\lambda_{\pm}) \approx \epsilon[\nu/(8\tilde{\delta}^2)]^{1/2} \ll \epsilon$ .

The second possibility to swing the modes is to choose  $b < 0$  in (14). Then  $c > a^2$ , and the eigenvalue  $\lambda_+$  is real (whereas  $\lambda_-$  is pure imaginary). There are two symmetrical regions of swinging in the parameter plane  $\tilde{\delta}, \tilde{\Delta}$ , which are located between the branches of hyperbolas (see Fig. 4):

$$\frac{\nu}{2(1-\tilde{\delta})} < \gamma < -\frac{\nu}{2(1+\tilde{\delta})}, \quad |\tilde{\delta}| > 1, \quad (36)$$

$$\frac{\nu}{2(1-\tilde{\delta})} < \gamma \quad \text{or} \quad \gamma < -\frac{\nu}{2(1+\tilde{\delta})}, \quad |\tilde{\delta}| \leq 1, \quad (37)$$

where  $\gamma \equiv \tilde{\Delta} - 3\tilde{\delta}$ . If  $|\tilde{\delta}| \leq 1$ , then we may parametrize  $\gamma$  as  $\gamma = -\nu\xi/4$ , where  $|\xi| \geq 1$ . In this case, the coefficient  $c$  has the following structure:  $c = \gamma^4(1 + 2\nu/\gamma^2 + c_2/\gamma^2 + \dots)$ , where  $c_2 \sim 1$ . Using formula (11) and expanding  $\sqrt{c}$  as  $\sqrt{c} = \gamma^2[1 + \nu/\gamma^2 + c_2/(2\gamma^2) - \nu^2/(2\gamma^4) + \dots]$ , we find, neglecting corrections of the order of  $\nu^{-1}$ ,

$$\lambda_+^2/\epsilon^2 = 1 - \tilde{\delta}^2 + 4(\tilde{\delta}\xi - 1)/\xi^2. \quad (38)$$

If  $|\xi| \gg 1$ , then we have the same expression for  $\lambda_+$  as in the case of a single resonance mode [7,8]:  $\lambda_+ = \sqrt{\epsilon^2 - \tilde{\delta}^2}$ . This means that the second mode goes out of resonance, if its detuning  $\Delta$  satisfies the inequality  $|\Delta| \gg \nu$ . It is interesting, however, that adjusting two detunings, one can achieve the maximal possible value  $\lambda_+^{(\max)} \approx \epsilon$  even for  $\delta \sim \epsilon$ , when the photon generation becomes impossible in the single-mode case: this happens for  $\xi_* = 2/\tilde{\delta}$ , i.e.,  $\gamma_* = -\nu/(2\tilde{\delta})$ .

For  $|\tilde{\delta}| > 1$  we write  $\gamma = -(1/2)\nu/(\tilde{\delta} + \chi)$ . Then the resonance exists for  $-1 < \chi < 1$ . Under the condition  $\nu \gg 1$ , the real eigenvalue  $\lambda_+$  can be written as (we neglect corrections of the relative order of  $\nu^{-1}$ )

$$\lambda_+ \approx \frac{\epsilon\nu\sqrt{1-\chi^2}}{\nu + 2\tilde{\delta}^2}. \quad (39)$$

Again, one can achieve the maximal possible value  $\lambda_+^{(\max)} \approx \epsilon$ , if  $\nu \gg \tilde{\delta}^2$ , but for  $\nu \ll \tilde{\delta}^2$  the maximal eigenvalue decreases as  $\epsilon\nu/(2\tilde{\delta}^2)$ .

The resonance width  $\Gamma_{\Delta}$  (the distance between two points of intersections between the vertical line  $\tilde{\delta} = \text{const}$  and the hyperbolas limiting the region of resonance in Fig. 4) equals  $\Gamma_{\Delta} = \nu/(\tilde{\delta}^2 - 1)$  (for  $|\tilde{\delta}| > 1$ ), so it decreases rapidly for  $|\tilde{\delta}| \gg 1$ . Similar

widths  $\Gamma_{\delta}^{(l,r)}$  (determined in an obvious way from the points of intersections between the horizontal line  $\tilde{\Delta} = \text{const}$  and the hyperbolas) are given by the expressions  $\Gamma_{\delta}^{(l,r)} = 1 \pm \sigma$ , where

$$\sigma = \frac{1}{6} \left[ \sqrt{(\tilde{\Delta} + 3)^2 + 6\nu} - \sqrt{(\tilde{\Delta} - 3)^2 + 6\nu} \right].$$

The sum of these widths does not depend on  $\tilde{\Delta}$ , and the smallest of them decreases as  $\Gamma_{\delta}^{(\text{small})} \approx 3\nu/(4\tilde{\Delta}^2)$  for  $|\tilde{\Delta}| \gg \nu$ .

Simple explicit solutions of the equations of motion can be found in the special case  $\tilde{\delta} = 1$ ,  $\gamma = -\nu/2$  (when the generation stops for the single mode, but the increment takes its maximal value for coupled two modes). In this case, the eigenvalues of matrix  $\mathcal{A}$  (10) are as follows:

$$\lambda_+ = R = 1 - \frac{2}{\nu} + \mathcal{O}(\nu^{-2}),$$

$$\lambda_- = iJ, \quad J = \frac{\nu}{2} + 1 + \mathcal{O}(\nu^{-2}).$$

With an accuracy up to terms of the order of  $\nu^{-1}$ , the solutions read as

$$x_1(t) = x_1(0) \left[ \left(1 - \frac{2}{\nu}\right) C_1^-(2R\tau; t) + \frac{2}{\nu} \cos \phi_1 \right] - p_1(0) \left[ \left(1 - \frac{2}{\nu}\right) S_1^-(2R\tau; t) - \frac{2}{\nu} \sin \phi_1 \right] + \frac{x_3(0)}{4\mu} [C_1^-(2R\tau; t) - \cos \phi_1] - \frac{p_3(0)}{12\mu} [S_1^-(2R\tau; t) + \sin \phi_1], \quad (40)$$

$$x_3(t) = x_3(0) \left[ \left(1 - \frac{2}{\nu}\right) \cos \phi_3 + \frac{2}{\nu} C_3^-(2R\tau; t) \right] + \frac{1}{3} p_3(0) \left[ \left(1 - \frac{2}{\nu}\right) \sin \phi_3 - \frac{2}{\nu} S_3^-(2R\tau; t) \right] + \frac{x_1(0)}{12\mu} [C_3^-(2R\tau; t) - \cos \phi_3] - \frac{p_1(0)}{12\mu} [S_3^-(2R\tau; t) + \sin \phi_3], \quad (41)$$

where the notation is the same as in the preceding section, and

$$\phi_k(\tau; t) = k\tilde{\omega}t - 2J\tau.$$

The mean energies of each mode depend on time as follows (we neglect corrections of the order of  $\nu^{-2}$ ,

because they are small; moreover, they increase with time slower than the preserved terms of the order of  $\nu^{-1}$ ):

$$\mathcal{E}_1 = \frac{\theta_1}{2} \left[ \left( 1 - \frac{4}{\nu} \right) \cosh(4R\tau) + \frac{4}{\nu} \psi(\tau) \right] + \frac{\theta_3}{\nu} [\cosh(4R\tau) + 1 - 2\psi(\tau)], \quad (42)$$

$$\mathcal{E}_3 = \frac{3}{2} \theta_3 \left[ 1 - \frac{4}{\nu} + \frac{4}{\nu} \psi(\tau) \right] + \frac{3\theta_1}{\nu} [\cosh(4R\tau) + 1 - 2\psi(\tau)], \quad (43)$$

where

$$\psi(\tau) \equiv \cosh(2R\tau) \cos(2J\tau).$$

We see that the energy of the first mode is almost the same that it would be in the case of a single mode under the condition of the strict resonance, i.e.,  $\theta_1 \times \cosh(4\tau)/2$ . The energy of the third mode is significantly less than the energy of the first mode, if  $\nu \gg 1$ . Therefore this regime of excitation can be named “asymmetrical”. For  $\tau > 1$ ,  $\mathcal{E}_3/\mathcal{E}_1 \approx 6/\nu$ . Note, however, that for the cubical cavity with  $\nu = 50/3$ , the energy of the third mode is only three times less than that of the first one. It is important, nonetheless, that the rates of increase of the energies of each mode are almost twice bigger than they were in the case of the strict resonance discussed in the preceding section.

For the invariant uncertainty products we obtain, with the same accuracy, almost identical expressions:

$$\mathcal{D}_{1,3} = \frac{\theta_{1,3}^2}{4} \left[ 1 - \frac{8}{\nu} + \frac{8}{\nu} \psi(\tau) \right] + \frac{\theta_1 \theta_3}{\nu} [\cosh(4R\tau) + 1 - 2\psi(\tau)]. \quad (44)$$

We see a drastic difference from the strict resonance case: now the IUP of each mode increases (asymptotically) exponentially with time. For  $\tau \gg 1$ , each mode appears in a highly mixed quantum state, with  $\mathcal{D}_1 = \mathcal{D}_3 = \theta_1 \theta_3 \exp(4R\tau)/(2\nu)$ .

The photon distributions in the first and the third modes turn out essentially different (and different from the distributions in the strict resonance case). We have shown in the preceding section, that the form of the PDF (oscillating or not) depends on the ratio  $\tilde{\mathcal{E}}/\mathcal{D}$ . For the first mode,  $\tilde{\mathcal{E}}_1/\mathcal{D}_1 \approx \nu/(2\theta_3)$  for  $\tau \gg 1$ . Consequently, the asymptotical PDF oscillates

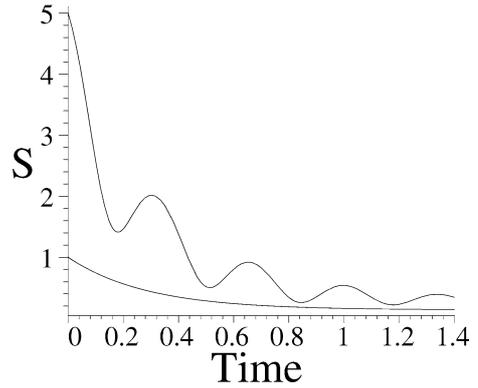


Fig. 5. The squeezing coefficient (26) versus “slow time”  $\tau$  (20) for the first mode in the case of “asymmetrical generation”, for  $\tilde{\delta} = 1$ ,  $\gamma = -\nu/2$ , and  $\nu = 50/3$ . The lower curve corresponds to the case  $\theta_1 = 1$  (vacuum initial state). The upper curve corresponds to the case  $\theta_1 = 5$ .

for the initial vacuum state and for initial thermal states with not very large mean numbers of photons (less than, approximately,  $\nu/4$ ). Also, the squeezing coefficient (26) tends to the finite asymptotical value  $s_1^\infty = \lim_{\tau \rightarrow \infty} (\mathcal{D}_1/\tilde{\mathcal{E}}_1) = 2\theta_3/\nu$  (typical dependences  $s_1(\tau)$  are given in Fig. 5). This resembles (qualitatively) the behavior of the squeezing coefficient in the one-dimensional cavity with an infinite number of coupled modes [10]. On the contrary, since  $\mathcal{D}_3/\tilde{\mathcal{E}}_3 \rightarrow \theta_3$  for  $\tau \rightarrow \infty$ , the final squeezing coefficient of the third mode coincides with the initial value:  $s_3(0) = s_3^\infty = \theta_3 \geq 1$ , with very small deviations from this value in the interval  $0 < \tau < \infty$ . The photon distribution function of the third mode does not oscillate, being close to the “quasigeometric” distribution (31).

## 5. Conclusion

We have studied the problem of photon creation due to the nonstationary (dynamical) Casimir effect in a three- or two-dimensional ideal cavity with an oscillating boundary, in the case when the spectrum of eigenfrequencies contains two frequencies, whose ratio is close to 3, and the boundary oscillates at the frequency close to the double lowest field eigenfrequency. We have calculated the mean energy, squeezing coefficient, invariant uncertainty product, and the photon distribution in each of two resonance modes, both under the strict and approximate resonance conditions.

We have shown that, due to a strong coupling between the resonance modes, it is possible to obtain the photon generation even for relatively large detunings between the frequencies of the wall and the field, compensating one detuning by another. However, the resonance width decreases with increase of detuning, as well as the increment of the exponential growth of the energy. We have considered both the vacuum and thermal initial states, having demonstrated not only an amplification of the number of created photons due to the initial thermal fluctuations, but also such effects as the “purity exchange” between the modes and the transformation of the initial smooth photon distribution to an oscillating distribution, typical for strongly squeezed states. Our results show that choosing a proper set of parameters of realistic three-dimensional cavities, one could facilitate observing the nonstationary Casimir effect, which is a challenge for experimentalists. However, a correct account of the influence of damping is still an unsolved problem.

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