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Shrinking quantum packets in one dimension

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Abstract

We give examples of “shrinking” free wave packets with zero initial velocity in one dimension. Considering different measures of spatial extension of wave packets, we show that degree of shrinking of “two-hump” packets corresponding to initial *even* coherent states is greater than for their mixed counterparts and for odd pure states.
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1. Introduction

Although propagation of quantum packets in free space was discussed for decades in numerous papers, beginning with [1], and in almost all textbooks on quantum mechanics, this subject is not exhausted yet. For recent few years, several new interesting effects in propagation and scattering (reflection) of packets of matter waves have been discovered [2–7]. Here we are studying peculiar features of the dynamics of one-dimensional packets, whose probability density initially has more than one maximum. Our interest to this problem is motivated by the recent article [7].

It is well known that, in the absense of external fields, initially localized packets spread unlimitedly with the course of time (for the most recent discussions see, e.g., [8]; the problem of controlling the shape of packets with the aid of external fields was considered, e.g., in [9]). However, it was shown recently [7] that certain “ring-shaped” packets in *two dimensions* experience initially some small “contraction”, and only after certain finite interval of time they begin to spread monotonously. More precisely, considering a free evolution in d space dimensions of the packet whose initial form depends only on radius as (we use dimensionless variables, assuming, in particular, $m = \hbar = 1$)

$$\psi_d(r;0) = \mathcal{N}_d r^2 \exp\left(-r^2\right), \tag{1}$$

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$t < t_*$ for $d = 2$, whereas for $d \geqslant 3$ it monotonously increases for $t > 0$. This effect of initial contraction of the packet was interpreted as a manifestation of an effective "quantum anticentrifugal potential" in two dimensions [10,11]. Unfortunately, the effect of contraction considered in [7] appeared to be extremely small, because the ratio of the minimal mean radius to its initial value equals 0.9978 for the initial packet (1) and 0.9953 for the best configuration among those considered in [7].

The aim of our Letter is to demonstrate that similar and even more pronounced effect of initial "shrinking" exists also in one space dimension (where no "anticentrifugal potential" can be used for explanation) for certain kinds of "two-hump" packets, whose simplest example is nothing but the *even coherent state*, introduced in [12] and frequently considered nowadays as a model of the "Schrödinger cat". Moreover, considering different quantitative measures of packet spreading, we have discovered that, in some sense, *odd coherent states* (which were studied in [2] in connection with the effect of *quantum deflection* of slow particles from mirrors) can also exhibit initial shrinking under certain conditions. In order to avoid confusion, we emphasize that we consider the packets described by means of *real* initial wave functions, so that the initial mean velocities or probability flux densities are equal to zero identically. Comparing free evolutions of the even and odd wave packets representing *pure* quantum states, and the evolution of the density matrix describing quantum mixture of two Gaussians, we show that the effect of "shrinking" of the initial packet is, as a matter of fact, a result of quantum interference.

The plan of the Letter is as follows. In Section 2 we consider the free evolution of even and odd coherent wave packets, comparing it with the evolution of their mixed state analogue. A quantitative measure of "shrinking" (expansion), identified with the average absolute value of coordinate is analyzed in Section 3. Two other measures of spatial extension, namely, "α-radii" (which are more sensitive to large or, on the contrary, small values of coordinate) and "β-extensions" (which, being translationally invariant, give the "proper volume" of the packet), are considered in Sections 4 and 5, respectively. Section 6 contains a brief discussion and conclusion.

2. Free evolution of even and odd coherent states

Making a "cut" of the multidimensional "ring-shaped" packet possessing the radial wave function (1) along, say, x-axis, one obtains a "two-hump" *even* function of x. Therefore, one can suppose, "in the first approximation", that if the effect of "shrinking" exists in one dimension, it should be related to "two-hump" even functions $\psi(x) = \psi(-x)$. To verify this idea, let us compare free evolutions of the even and odd coherent states of a harmonic oscillator with unit frequency, mass, and Planck's constant [12] (normalized on the whole axis $-\infty < x < \infty$), after the binding harmonic potential has been switched off at $t = 0$:

$$\psi^{(\pm)}(x,t) = \mathcal{N}_\pm\big[\varepsilon(t)\big]^{-1/2}\exp\biggl(-\frac{x^2+x_c^2}{2\varepsilon}\biggr) \times \begin{Bmatrix}\cosh\\ \sinh\end{Bmatrix}\left(\frac{xx_c}{\varepsilon(t)}\right), \tag{2}$$

where

$$\varepsilon(t) = 1 + it, \tag{3}$$

$$\mathcal{N}_\pm = \sqrt{2}\,\pi^{-1/4}\big[1 \pm \exp\big(-x_c^2\big)\big]^{-1/2}. \tag{4}$$

The initial wave functions at $t = 0$ are obtained simply by putting $\varepsilon = 1$. In order to ensure the absense of any initial fluxes of probability, parameter x_c must be *real*, and we choose it positive for convenience.[2] The probability density equals

$$\big|\psi^{(\pm)}(x,t)\big|^2 = \frac{\mathcal{N}_\pm^2}{2|\varepsilon(t)|}\exp\biggl(-\frac{x^2+x_c^2}{|\varepsilon(t)|^2}\biggr) \times\left[\cosh\biggl(\frac{2xx_c}{|\varepsilon(t)|^2}\biggr) \pm \cos\biggl(\frac{2xx_ct}{|\varepsilon(t)|^2}\biggr)\right]. \tag{5}$$

The odd probability distribution always has two peaks, because it equals zero for $x = 0$. The even probability

[2] Taking the initial wave function in the form $\mathcal{N}\exp[-\frac{1}{2}(1+i\kappa)x^2]$ with $\kappa > 0$, one can easily see that such a packet (named *correlated coherent state* in [13] due to nonzero value of the covariance $\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle$, and "contractive state" in [14,15]) obviously shrinks at small values of time. However, in such a case a nonzero initial probability flux density is directed to the origin, so the effect of shrinking becomes trivial. For this reason we confine ourselves to the case of $\kappa = 0$. Another trivial example is a superposition of two real localized packets with phase factors $\exp(\pm ip_0x)$, when both the peaks move in the direction to the origin, if $p_0 \neq 0$.

distribution has initially two peaks (whose positions are determined by the equation $\tanh(xx_c) = x/x_c$) under the condition $x_c > 1$ (if $x_c \gg 1$, the peaks are located nearby the points $\pm x_c$). The evolution of packets given by (5) with $x_c = \sqrt{2}$ (this choice will be clear in Section 3) is shown in Figs. 1 and 2. The value $t_1 = 1/\sqrt{3}$ (or $t_1 = \sqrt{(x_c^2 - 1)/(x_c^2 + 1)}$ for an arbitrary x_c) corresponds to the instant when the second spatial derivative of the even probability density at the point $x = 0$ changes its sign. At the instant $t_2 = \sqrt{3}$ (or $t_2 = \sqrt{2x_c^2 - 1}$ in a generic case) the function $|\psi^{(+)}(0,t)|^2$ attains its maximum.

It is worth comparing *coherent* packets with a quantum *mixture* of two Gaussians, described by the time-dependent density matrix (see Fig. 3)

$$\rho(x,x',t) = \exp\left(-\frac{x^2\varepsilon^* + x'^2\varepsilon + 2x_c^2}{2|\varepsilon(t)|^2}\right) \times \cosh\left[\frac{x_c}{|\varepsilon(t)|^2}\left(x\varepsilon^* + x'\varepsilon\right)\right]\left(|\varepsilon|\sqrt{\pi}\right)^{-1}. \tag{6}$$

One can easily verify that the probability density and other quantities characterizing the mixed state (6), can be obtained from the corresponding expressions for its even/odd partners by means of a simple recipe: one should replace $\mathcal{N}_\pm^2$ by $2\pi^{-1/2}$ and delete the terms with $\pm$ sign. For the *Wigner function*

$$W(x,p) = \int \psi(x + y/2)\psi^*(x - y/2)\exp(-ipy)\,dy$$

we obtain

$$W_\pm(x,p,t) = \mathcal{N}_\pm^2\sqrt{\pi}\exp\left[-p^2 - (x - pt)^2\right] \times \left\{e^{-x_c^2}\cosh\left[2x_c(x - pt)\right] \pm \cos(2px_c)\right\}. \tag{7}$$

Looking at Fig. 1, one has a clear impression that at the instant $t = \sqrt{3}$, the packet is "narrower" than it was initially, especially if one does not pay attention to the "wings" of the distribution at $|x| > 2.5$. On the other hand, from Fig. 2 it seems obvious that the odd packet expands monotonously. An additional information on the behaviour of the packets is contained in the probability flux density

$$J(x) = \mathrm{Im}\left[\psi_x(x)\psi^*(x)\right] = \mathrm{Im}\left[\rho_x(x,x')\right]\big|_{x'=x}. \tag{8}$$

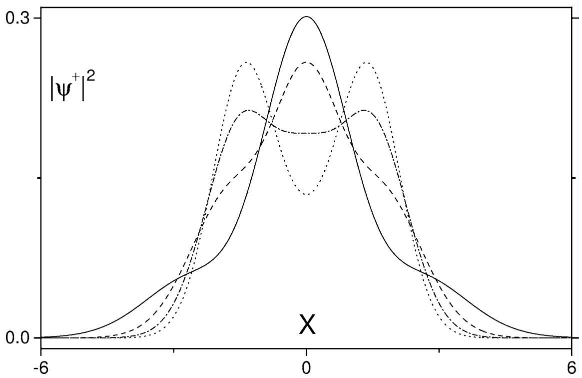

Fig. 1. The probability densities of the even coherent packet with $x_c = 2^{1/2}$ at different instants of time: $t = 0$ (dotted curve with two sharp peaks), $t = 3^{-1/2}$ (dash-dotted curve), $t = 1$ (dashed curve), and $t = 3^{1/2}$ (solid curve).

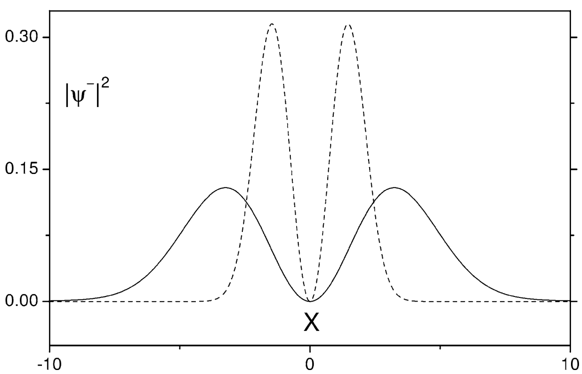

Fig. 2. The probability densities of the odd coherent packet with $x_c = 2^{1/2}$ at different instants of time: $t = 0$ (dashed curve with high peaks) and $t = 4$ (solid curve).

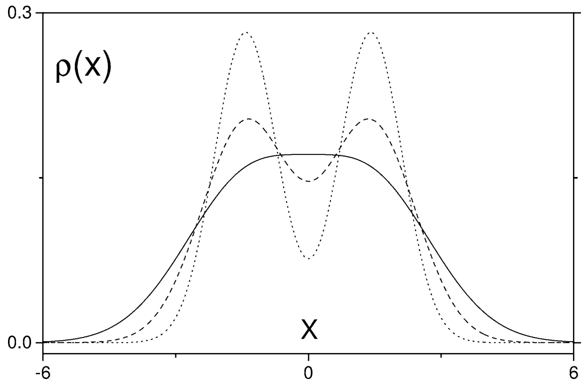

Fig. 3. The probability densities of the totally mixed packet with $x_c = 2^{1/2}$ at different instants of time: $t = 0$ (dotted curve with high peaks), $t = 1$ (dashed curve), and $t = 3^{1/2}$ (solid curve).

For the state (2) we obtain

$$J^{(\pm)}(x,t) = \frac{\mathcal{N}_\pm^2 xt}{2|\varepsilon(t)|^3} \exp\left(-\frac{x^2+x_c^2}{|\varepsilon(t)|^2}\right) \times \left\{ \cosh\left(\frac{2xx_c}{|\varepsilon(t)|^2}\right) - \frac{x_c}{x}\sinh\left(\frac{2xx_c}{|\varepsilon(t)|^2}\right) \pm \cos\left(\frac{2xx_c t}{|\varepsilon(t)|^2}\right) \mp \frac{x_c}{xt}\sin\left(\frac{2xx_c t}{|\varepsilon(t)|^2}\right)\right\}. \quad (9)$$

In the case of the mixed state (6) one should use the recipe given above. For $t \ll 1$ we have

$$J^{(\pm)}(x) = \frac{xt}{2}\mathcal{N}_\pm^2 \exp\left(-x^2 - x_c^2\right) \times \left[\pm\left(1-2x_c^2\right) + \cosh(2xx_c) - \frac{x_c}{x}\sinh(2xx_c)\right],$$

and one can easily check that in this limit, $J^{(+)}(x)$ is negative (for x positive) in the interval $0 < x \leqslant x_c$, if $x_c > 1$, whereas $J^{(-)}(x)$ and $J^{(\mathrm{mix})}(x)$ are positive for $x = x_c$. Fig. 4 also indicates that the probability density in the even packet flows initially mainly to the centre, whereas the main tendency in the odd packet is to move outside. But how to quantify “shrinking” or “expansion”?

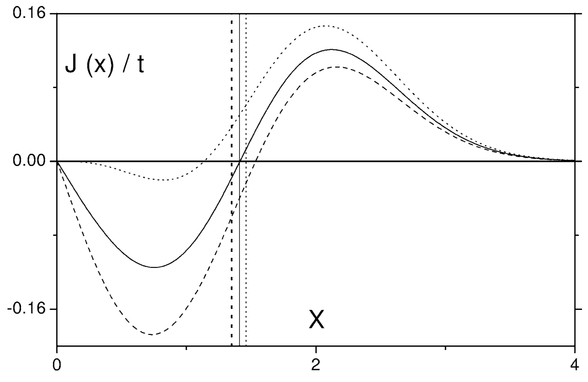

Fig. 4. The initial time derivative of the probability density flux, $J(x)/t$, for the even (dashed line), odd (dotted line) and mixed (solid line) packets in the case of $x_c = 2^{1/2}$. Vertical lines show the positions of initial maxima for each packet: $x_{\max}^{(+)} = 1$.

holds for $t > 1.8$ (these numerical values correspond to $x_c = \sqrt{2}$).

The coincidence of the condition $x_c > 1$ with the condition of existence of two peaks in the initial probability distribution is not accidental. To show this, let us calculate the second derivative of the integral (10) with respect to time, taking into account the Schrödinger equation $\psi_t = (i/2)\psi_{xx}$ and the boundary condition $\psi_x|_{x=0} = 0$, which must be fulfilled for any even function of x. After simple algebra (including integrations by parts) we obtain the equation

$$\frac{d^2\bar{x}_+}{dt^2} = -\operatorname{Re}\big(\psi^*\psi_{xx}\big)\big|_{x=0}, \tag{15}$$

where the right-hand side can be interpreted as an "effective quantum acceleration" caused by the presence of a specific "impenetrable boundary" at the point $x = 0$, characterized by the boundary condition $\psi_x|_{x=0} = 0$, which results in zero flux density at $x = 0$ (as well as in the case of $\psi(0) = 0$). Using the dimensional variables, the right-hand side of (15) should be multiplied by $(\hbar/m)^2$. We see that the effective acceleration is negative, if the second derivative of the wave function at $x = 0$ has the same sign as the wave function itself. But if $x_c < 1$, then the initial (real) wave function has a single *maximum* at the origin, so that the product $\psi^*\psi_{xx}|_{x=0}$ is negative, and the effective force is repulsive. On the contrary, for $x_c > 1$ the initial wave function has *minimum* at the origin, and the effective acceleration is negative (for $t = 0$ and in some interval $0 < t < t_*$).

On the other hand, for all packets described by *odd* wave functions, $\psi(x) = -\psi(-x)$, the mean absolute coordinate increases monotonously for $t > 0$. For example, the first terms of the Taylor expansion of (12) read

$$\bar{x}_- = \bar{x}_-(0) + \frac{t^2}{2}\mathcal{N}_-^2 x_c^2 e^{-x_c^2} + \mathcal{O}\big(t^4\big). \tag{16}$$

The mean absolute coordinate in the mixed packet also increases monotonously, although the initial "acceleration" in the mixed state is more than twice less than in the pure odd state with the same value of x_c:

$$\bar{x}_{\text{mix}}(t) = \bar{x}(0) + \frac{t^2 x_c^2}{2\sqrt{\pi}} e^{-x_c^2} + \cdots.$$

There are at least two "explanations" of the difference in the behaviour of even and odd packets. One of them is that the odd packet can be considered, in certain sense, as a one-dimensional analogue of multidimensional states with *nonzero angular momentum* (as soon as transformation $x \to -x$ can be considered as an analogue of the rotation by angle π), for which an "effective radial centrifugal force" is always positive. Another explanation is that odd packets are equivalent to the states on the semiaxis $x > 0$, confined by the usual impenetrable wall, represented by the boundary condition $\psi(0) = 0$. In this case, the right-hand side of Eq. (15) equals [2,16] $(\hbar/m)^2|\psi_x|^2|_{x=0} \geqslant 0$ (in dimensional variables and for the wave functions normalized on the whole axis $-\infty < x < \infty$). Possible manifestations of such a "quantum repulsive force" for ultracold particles were discussed in [2,3].

The first derivative of $\bar{x}_-(t)$ over time equals

$$\frac{d\bar{x}_-}{dt} = \frac{\sqrt{\pi}}{2}\mathcal{N}_-^2 x_c t \operatorname{erfi}\left(\frac{x_c t}{|\varepsilon|}\right) e^{-x_c^2}, \tag{17}$$

whereas

$$\frac{d\bar{x}_+}{dt} = \mathcal{N}_+^2 \exp\left(-\frac{x_c^2}{|\varepsilon(t)|^2}\right) \times \left[\frac{t}{|\varepsilon|} - \frac{\sqrt{\pi}}{2} x_c \operatorname{erfi}\left(\frac{x_c t}{|\varepsilon|}\right) \exp\left(-\frac{t^2 x_c^2}{|\varepsilon(t)|^2}\right)\right]. \tag{18}$$

The right-hand side of (18) is negative for small values of time variable t (provided $x_c > 1$), but for $t \gg 1$ it tends to the constant positive value

$$\left.\frac{d\bar{x}_+}{dt}\right|_{t=\infty} = \mathcal{N}_+^2\left[1 - \frac{\sqrt{\pi}}{2} x_c \operatorname{erfi}(x_c) e^{-x_c^2}\right].$$

If $x_c \gg 1$, then the asymptotical formula [17]

$$\operatorname{erfi}(x \gg 1) \approx \big(\sqrt{\pi}\,x\big)^{-1} e^{x^2} \tag{19}$$

leads to identical limit values $(d\bar{x}_\pm/dt)|_{t=\infty} = \pi^{-1/2}$. In the case of ideal wall at the origin described by the boundary condition $\psi(0) = 0$ (i.e., for odd packets) this result was obtained in [2], and its actual independence of the concrete form of an *impenetrable* barrier was explained in [3].

Not every initial even packet with two or more peaks exhibits the effect of shrinking (in terms of the parameter $\bar{x}$). For example, any "Fock-like" packet

$$\psi_n(x,t) = \frac{\mathcal{N}_n(\varepsilon^*)^{n/2}}{\varepsilon^{(n+1)/2}} \exp\left(-\frac{x^2}{2\varepsilon}\right) H_n\left(\frac{x}{|\varepsilon|}\right) \tag{20}$$

expands monotonously, $\bar{x}(t) \sim |\varepsilon(t)|$, because function (20) satisfies the stationary Schrödinger equation $-\psi_n'' + x^2\psi_n = 2E_n\psi_n$ with positive eigenvalues E_n ($H_n(z)$ is the Hermite polynomial). Consequently, for any (even) solution of this type, the second derivative ψ'' at $x = 0$ has opposite sign with respect to the wave function ψ, resulting in positive effective acceleration (15). The same is true for any state which was initially a positive energy eigenstate of some potential $V(x)$ possessing the property $V(0) = 0$. However, for *negative energy eigenstates* (for example, in symmetrical two-well potentials with a local maximum at $x = 0$ and absolute minima at some $|x_m| > 0$) the initial effective acceleration (15) is negative. This observation could be used to try to find packets with smaller minimal value of the ratio $\bar{x}(t)/\bar{x}(0)$ than for the even coherent state packets.

It is worth noting that the wave function (20) does not contain any parameter (except for n). For this reason, the form of the probability density does not change during the evolution, which is reduced to the scale transformation $x \to x/|\varepsilon(t)|$. On the contrary, all packets exhibiting shrinking are not scale invariant, since they depend on some extra parameters, such as x_c.

4. "α-measures" of spreading

The quantity (10) is, to certain extent, a *biased* measure of packet spreading, because it is more sensitive to higher values of $|x|$. For example, although the central part of the packet shown in Fig. 1 for $t = \sqrt{3}$ is obviously narrower than the packet at $t = 1$, the ratio $\bar{x}_+(t)/\bar{x}_+(0)$ for $t = \sqrt{3}$ is slightly higher than for $t = 1$, due to a bigger contribution of the "wings" of the probability density (although both ratios are less than 1). Moreover, if one prefers to use the *mean square radius*,

$$\bar{x}^{(2)} \equiv \left(\int x^2 \left|\psi(x)\right|^2 dx\right)^{1/2},$$

then *any* packet with initial real wave function (i.e., with zero initial momentum and zero initial covariance $\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle$) spreads with time monotonously, due to the exact solution of the Heisenberg equations of motion for a free particle,

$$\langle \hat{x}^2(t)\rangle = \langle \hat{x}^2(0)\rangle + \langle \hat{p}^2\rangle t^2, \quad \langle \hat{p}^2\rangle = \text{const}.$$

We introduce a family of "α-radii"

$$\bar{x}^{(\alpha)} \equiv \left(\int |x|^\alpha \left|\psi(x)\right|^2 dx\right)^{1/\alpha}. \tag{21}$$

For the state (2), the integral in (21) can be expressed through the parabolic cylinder functions or confluent hypergeometric function $\Phi(a; c; z)$ [17]:

$$\bar{x}_\pm^{(\alpha)}(t) = \left|\varepsilon(t)\right| \left[\frac{1}{2}\mathcal{N}_\pm^2 \Gamma\left(\frac{1+\alpha}{2}\right)\right]^{1/\alpha} \times \left[\Phi\left(-\frac{\alpha}{2}; \frac{1}{2}; -\frac{x_c^2}{|\varepsilon|^2}\right) \pm e^{-x_c^2}\Phi\left(-\frac{\alpha}{2}; \frac{1}{2}; \frac{x_c^2 t^2}{|\varepsilon|^2}\right)\right]^{1/\alpha}. \tag{22}$$

To expand the right-hand side of (22) in the Taylor series with respect to the time variable, we use the following relations for the function $\Phi(a; c; z)$ [17]:

$$\frac{d}{dz}\Phi(a; c; z) = \frac{a}{c}\Phi(a+1; c+1; z),$$
$$\Phi(a; c; -z) = e^{-z}\Phi(c-a; c; z),$$
$$c\Phi(a-1; c; z) - c\Phi(a; c; z) = z\Phi(a; c+1; z).$$

After some algebra we obtain the expression

$$\bar{x}_\pm^{(\alpha)} = \left[\frac{\Gamma\left(\frac{1+\alpha}{2}\right)A_\pm}{\sqrt{\pi}(e^{x_c^2} \pm 1)}\right]^{1/\alpha}\left[1 + \frac{B_\pm}{A_\pm}t^2 + \cdots\right],$$

where

$$A_\pm = \Phi\left(\frac{\alpha+1}{2}; \frac{1}{2}; x_c^2\right) \pm 1,$$
$$B_\pm = \frac{1}{2}\Phi\left(\frac{\alpha-1}{2}; \frac{1}{2}; x_c^2\right) \pm \frac{1}{2} \mp x_c^2.$$

The α-radius initially decreases with time provided $B_\pm/A_\pm < 0$. Using the asymptotical formula

$$\Phi(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}, \quad z \to +\infty,$$

one can see that for $x_c \gg 1$ (and $\alpha \neq 1$),

$$B_\pm/A_\pm \sim \Gamma\left(\frac{\alpha+1}{2}\right)\left[2x_c\Gamma\left(\frac{\alpha-1}{2}\right)\right]^{-1}.$$

Consequently, in terms of the α-radii, all packets (even, odd, and mixed) with well separated peaks exhibit initial shrinking for $\alpha < 1$, which is quite pronounced for the values of α close to -1: see Fig. 5. The explanation of such a behaviour becomes clear, if one looks at Fig. 6: each initial peak expands almost symmetrically from its initial maximum, therefore we observe an increase of probability density in the region between initial peaks. Since α-measures with $\alpha < 1$ are more sensitive to the contributions from this "inner" region, α-radii exhibit rapid decrease at the initial stage of evolution.

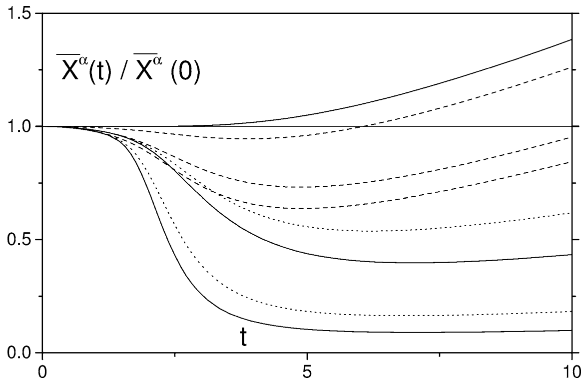

Fig. 5. The time dependence of normalized "α-radii" $\bar{x}^{(\alpha)}(t)/\bar{x}^{(\alpha)}(0)$ for the even, odd, and mixed packets in the case of $x_c = 5$. Solid lines, from top to bottom: $\alpha = 1, -1/2, -9/10$. Dashed lines, from top to bottom: $\alpha = 1/2, -1/2, -9/10$. Dotted lines, from top to bottom: $\alpha = -1/2, -9/10$. For $\alpha \geqslant 1/2$, the curves corresponding to different packets turn out very close, with the same order of appearance as in the cases shown: odd, mixed, even (from top to bottom). The initial values $\bar{x}^{(\alpha)}(0)$ practically do not depend on the kind of packet, and only slightly depend on α: $\bar{x}^{(1)}(0) = 5.00$ and $\bar{x}^{(-9/10)}(0) = 4.90$.

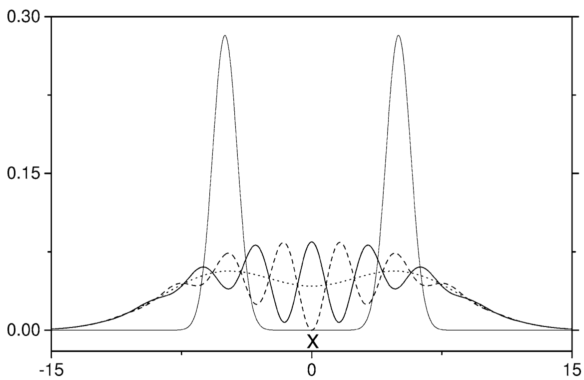

Fig. 6. Probability densities of different kinds of packets with $x_c = 5$. The fine solid curve with two peaks gives the initial distribution at $t = 0$, when all packets practically coincide. Other curves correspond to the moment $t = 5$: solid curve (with maximum at $x = 0$)—even packet, dashed curve (with zero minimum at $x = 0$)—-odd packet, and intermediate dotted curve—mixed packet.

5. "β-extensions" of packets

Different *translationally invariant* measures of spatial (temporal) extensions of wave packets (signals), which do not give preference to large or small values of coordinates, were introduced by several authors [18–20] (for review see [21]). Following [19], we define "β-extension" of the normalized probability distribution $\rho(x)$ (where $\rho(x) = \rho(x, x)$ or $\rho(x) = |\psi(x)|^2$) as

$$\mathcal{E}^{(\beta)} = \left(\int \left[\rho(x)\right]^{\beta} dx\right)^{1/(1-\beta)}. \tag{23}$$

A reasonability of such a definition can be checked in the simplest example of the uniform distribution $\rho(x) = a^{-1}$ at some interval (not necessarily single-connected) of the length a: in this case, $\mathcal{E}^{(\beta)} \equiv a$ for any β. Taking the limit $\beta \to 1$, one arrives at the "entropic measure" of extension,

$$\mathcal{E}^{(1)} \equiv \exp(S_x), \quad S_x = -\int \rho(x) \ln\left[\rho(x)\right] dx,$$

introduced in [18] (see also [22]). However, from the point of view of simplicity of calculations, the choice $\beta = 2$ is better:

$$\mathcal{E}^{(2)} = \left(\int \left[\rho(x)\right]^{2} dx\right)^{-1}. \tag{24}$$

For the distribution (5) we find

$$\mathcal{E}_{\pm}^{(2)} = \frac{8\sqrt{2}|\varepsilon(t)|}{\sqrt{\pi}\,\mathcal{N}_{\pm}^{4}} \times \left\{1 + e^{-2x_c^2} + 2\exp\left(-\frac{2x_c^2}{|\varepsilon|^2}\right) \pm 4\exp\left[-\frac{x_c^2(3+t^2)}{2(1+t^2)}\right]\cos\left(\frac{x_c^2 t}{|\varepsilon|^2}\right)\right\}^{-1}. \tag{25}$$

For the mixed initial state (6) one should delete the last term with $\pm$ sign and replace $\mathcal{N}_{\pm}^{4}$ by $4/\pi$.

Another simple expression for the extension arises in the case of $\beta = \infty$, because one can verify that

$$\mathcal{E}^{(\infty)} \equiv \left(\max\left[\rho(x)\right]\right)^{-1}. \tag{26}$$

The time dependences of different "β-measures" are shown in Fig. 7. For $\beta = 2$, no packets exhibit shrinking, although the time dependence is not monotonous in the case of even packets. However, even packets clearly demonstrate an intermediate decrease of the limiting ratio $\overline{\mathcal{E}}^{(\infty)}(t) \equiv \mathcal{E}^{(\infty)}(t)/\mathcal{E}^{(\infty)}(0)$ below the initial level, in full accordance with Fig. 1. The maximum of the solid curve in Fig. 7 is achieved at the moment $t_{\max} \approx 0.65$, when the height of the central peak at $x = 0$ becomes equal the heights of symmetrical lateral peaks. The minimum of the solid curve, $\overline{\mathcal{E}}^{(\infty)}_{\min} \approx 0.856$, corresponds to the moment $t_2 = \sqrt{3}$ when the probability density at the origin, $\rho(0, t)$, attains its maximum. Assuming that the position of maximum at $t = 0$ coincides with x_c (this approximation is rather good already for $x_c = \sqrt{2}$, being much better for $x_c \gg 1$), one can obtain an approximate formula $\overline{\mathcal{E}}^{(\infty)}_{\min} \approx x_c\sqrt{e/8}$. Therefore, shrinking of even packets with respect to the $\mathcal{E}^{(\infty)}$-measure exists for the values of x_c belonging to the interval $1 < x_c < \sqrt{8/e} \approx 1.7$.

For mixed packets, the expression for t_1^{mix} coincides with the expression for t_2 (which is the same as for even packets). Therefore, mixed packets do not shrink with respect to $\mathcal{E}^{(\infty)}$-measure, as well as odd ones (which expand more rapidly), in accordance with Figs. 2 and 3.

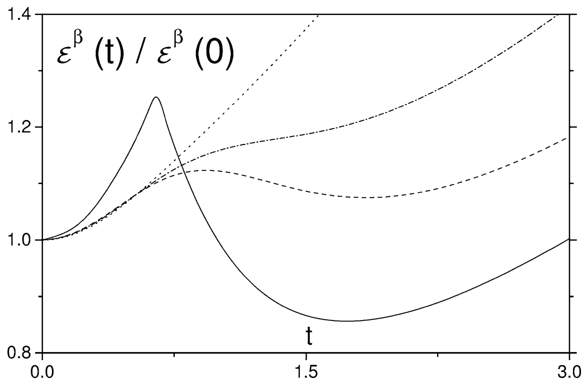

Fig. 7. Time dependences of normalized "β-extensions" $\mathcal{E}^{(\beta)}(t)/\mathcal{E}^{(\beta)}(0)$ for different initial packets with $x_c = 2^{1/2}$. Three curves above the unity level correspond to $\beta = 2$. Their order from top to bottom is as follows: odd packet (dots), mixed packet (dash-dots), even packet (dashes). The solid curve corresponds to the limiting ratio $\mathcal{E}^{(\infty)}(t)/\mathcal{E}^{(\infty)}(0)$ for even packets.

6. Conclusion

We have demonstrated that there exist families of one-dimensional quantum packets exhibiting the property of initial "shrinking" in the process of free evolution, although initial mean values of momentum, covariances, or probability flux density are equal to zero. All such packets have more than one peak at the initial moment. The degree of shrinking depends on the chosen quantitative measure of the spatial extension. The effect is more pronounced for pure even packets than for odd or mixed ones, moreover, in many cases it does not exist for the last two kinds of packets. We consider this difference as a manifestation of quantum interference.

It was supposed in [7] that the existence of (much smaller) shrinking effect for two-dimensional isotropic radial packets and its absence in the 3D case could be connected with a greater degree of "nonclassicality" of the two-dimensional packets, characterized by the relative weight of the negative part of the Wigner function. However, our one-dimensional examples do not confirm this conjecture. Calculating integrals over positive and negative parts of the Wigner function,

$$V_\pm = \frac{1}{2}\int \left[W(q,p) \pm \left|W(q,p)\right|\right]\frac{dq\,dp}{2\pi},$$

(using the normalization $V_+ + V_- = 1$), we have obtained for the function (7) with $x_c = \sqrt{2}$ the following values:

$$V_-^{\text{ev}} = -0.104, \qquad V_+^{\text{ev}} = 1.104,$$
$$V_-^{\text{od}} = -0.217, \qquad V_+^{\text{od}} = 1.217.$$

According to these numbers, the odd state is "more nonclassical", than the even one. However, the shrinking effect is observed not for the odd, but for the even packet.

Applications of the new measures of spatial extension to multidimensional packets will be considered in another publication.

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References

[1] C.G. Darwin, Proc. Roy. Soc. A 117 (1927) 258.

[2] V.V. Dodonov, M.A. Andreata, Phys. Lett. A 275 (2000) 173; V.V. Dodonov, M.A. Andreata, Laser Phys. 12 (2002) 57.

[3] M.A. Andreata, V.V. Dodonov, J. Phys. A 35 (2002) 8373.

[4] G. Kälbermann, Phys. Rev. A 60 (1999) 2573; G. Kälbermann, J. Phys. A 34 (2001) 3841; G. Kälbermann, J. Phys. A 34 (2001) 6465; G. Kälbermann, J. Phys. A 35 (2002) 4599; G. Kälbermann, J. Phys. A 35 (2002) 9829.

[5] J.P. Keating, J.M. Robbins, J. Phys. A 34 (2001) 807.

[6] M.A. Efremov, M.V. Fedorov, Laser Phys. 12 (2002) 786; M.A. Efremov, M.V. Fedorov, Phys. Rev. A 65 (2002) 052725.

[7] I. Bialynicki-Birula, M.A. Cirone, J.P. Dahl, M. Fedorov, W.P. Schleich, Phys. Rev. Lett. 89 (2002) 060404.

[8] G.W. Ford, R.F. O'Connell, Am. J. Phys. 70 (2002) 319.

[9] F. Delgado, B. Mielnik, J. Phys. A 31 (1998) 309.

[10] M.A. Cirone, K. Rzążewski, W.P. Schleich, F. Straub, J.A. Wheeler, Phys. Rev. A 65 (2002) 022101.

[11] I. Bialynicki-Birula, M.A. Cirone, J.P. Dahl, R.F. O'Connell, W.P. Schleich, J. Opt. B 4 (2002) S393.

[12] V.V. Dodonov, I.A. Malkin, V.I. Man'ko, Physica 72 (1974) 597.

[13] V.V. Dodonov, E.V. Kurmyshev, V.I. Man'ko, Phys. Lett. A 79 (1980) 150.

[14] H.P. Yuen, Phys. Rev. Lett. 51 (1983) 719.

[15] P. Storey, T. Sleator, M. Collett, D. Walls, Phys. Rev. A 49 (1994) 2322.

[16] B.S. Kay, J. Phys. A 14 (1981) 155.

[17] Bateman Manuscript Project, in: A. Erdélyi (Ed.), Higher Transcendental Functions, McGraw-Hill, New York, 1953.

[18] D.A. Bochvar, I.V. Stankevich, A.L. Chistyakov, Dokl. Akad. Nauk SSSR 135 (1960) 1095; D.A. Bochvar, I.V. Stankevich, A.L. Chistyakov, Sov. Phys. Dokl. 5 (1961) 1271.

[19] M. Zakai, Inform. Control 3 (1960) 101.

[20] J. Hilgevoord, J.B.M. Uffink, Phys. Lett. A 95 (1983) 474.

[21] V.V. Dodonov, V.I. Man'ko, in: M.A. Markov (Ed.), Invariants and the Evolution of Nonstationary Quantum Systems, in: Proceedings of Lebedev Physics Inst., Vol. 183, Nova Science, Commack, 1989, p. 3.

[22] M. Grabowski, J. Math. Phys. 22 (1981) 303.