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Is there a lower bound energy in the harmonic oscillator interacting with a heat bath?

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Abstract

In this Letter we investigate the lower bound energy of the usual Hamiltonian employed in Quantum Optics to model the interaction between a harmonic oscillator and a reservoir without the rotating wave approximation. We show that this model has serious inconsistencies and then we discuss the origin of these inconsistencies.

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1. Introduction

The interaction of a single quantum system with a heat bath has become a standard procedure to study many phenomena in Quantum Optics [1–9]. Traditional examples are the cases when the single quantum system is a harmonic oscillator or a two-level atom coupled linearly to a heat bath modelled as an infinite set of harmonic oscillators [1–9] (linear-coupling model only consider an interaction of the form $x\sum_j m_j\omega_j^2 q_j$). This is an useful model employed to study theoretical and experimentally the decoherence of superposition of quantum states.

The most used model in Quantum Optics is the Jaynes–Cummings model, which describes the interaction between two single quantum systems: one of them being a harmonic oscillator (a cavity mode, for example) and the other a two level atom. The Jaynes–Cummings model has been used in Quantum Optics to describe both the interaction of a single two level atom and a cavity field and the dynamics of a laser cooled ion [10].

A good physical model gives clear explanations of huge experimental and theoretical facts and effects, and provides improved understanding on their applica-

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bility and restrictions. On the other hand, a not so good model is characterized by some impractical theoretical results and predictions. In any case, in order to avoid misinterpretations and misleading results, it is necessary to have careful parsimony when approximations are made in the models, and it has to be stated explicitly.

In both models mentioned above the Rotating Wave Approximation (RWA) is invoked to yield analytical results. However, Ford and O'Connell [11] have showed that the first model above, in the RWA, has not a lower bound energy; thus, one can extract energy of the single harmonic oscillator, even in the weak-coupling regime, only by making it going into an arbitrarily lower-energy state.

In this Letter we investigate the lower bound energy of a single quantum system (modelled as a single harmonic oscillator) coupled to a heat bath (modelled as an infinite set of harmonic oscillator), by employing an usual Hamiltonian in textbooks of quantum optics. We show that this Hamiltonian suffers from the same defect pointed out in Ref. [11] (i.e., its spectrum has no lower bound) even when the non-rotating terms are taking into account, and then we clarify where this Hamiltonian comes from.

2. The Hamiltonian without rotating wave approximation

In the absence of RWA, the Hamiltonian describing the field and reservoir systems reads [4–9,12–14]

$$\widehat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_j \hbar\omega_j\left(\hat{b}_j^\dagger\hat{b}_j + \frac{1}{2}\right) + \sum_j\left(\lambda_j\hat{b}_j + \lambda_j^*\hat{b}_j^\dagger\right)\left(\hat{a} + \hat{a}^\dagger\right), \tag{1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the field mode, respectively, obeying $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{b}_j$ and $\hat{b}_j^\dagger$ are the creation and annihilation operators of the jth oscillator of the reservoir having frequency ω_j and obeying the commutation relation $[\hat{b}_j, \hat{b}_k^\dagger] = \delta_{jk}$, ω_0 is the frequency of the single mode, and λ_j is the coupling constant field–reservoir.

In order to survey the lower bound energy in the Hamiltonian of Eq. (1), we assume an initial normalized state consisting of a product of coherent states of the form (as was implemented by Ford and O'Connell [11]):

$$|\Psi\rangle = |\alpha\rangle\prod_j|\beta_j\rangle. \tag{2}$$

Taking the expectation value of the Hamiltonian $\langle\Psi|\widehat{H}|\Psi\rangle$ in Eq. (1), we obtain

$$\langle\Psi|\widehat{H}|\Psi\rangle = \hbar\omega_0\left(|\alpha|^2 + \frac{1}{2}\right) + \sum_j\hbar\omega_j\left(|\beta|_j + \frac{1}{2}\right) + \sum_j\left(\alpha + \alpha^*\right)\left(\lambda_j\beta_j + \lambda_j^*\beta_j^*\right). \tag{3}$$

For a given value of α, we can minimize the Eq. (3) if we take β_j as

$$\beta_j = -\frac{\lambda_j^*(\alpha + \alpha^*)}{\hbar\omega_j}. \tag{4}$$

By substituting this value of β_j in Eq. (3) it results

$$\langle\Psi|\widehat{H}|\Psi\rangle\big|_{\min} = \hbar\omega_0\left(|\alpha|^2 + \frac{1}{2}\right) + \sum_j\hbar\omega_j - \sum_j\left(\alpha + \alpha^*\right)^2\frac{|\lambda|^2}{\hbar\omega_j}. \tag{5}$$

The right-hand side of Eq. (5) can be rearranged in the form

$$\left(\hbar\omega_0 - 2\sum_j\frac{|\lambda_j|^2}{\hbar\omega_j}\right)|\alpha|^2 - \sum_j\left(\alpha^2 + \alpha^{*2}\right)\frac{|\lambda_j|^2}{\hbar\omega_j} + \sum_j\hbar\omega_j. \tag{6}$$

As it seems, by comparing the Eq. (6) with Eq. (8) of Ref. [11], we can apply the same reasoning made there to the first term of Eq. (6), to ascertain that there is no lower bound to the energy of the complete Hamiltonian Eq. (1), i.e., to the model without RWA, and no matter if the weak or strong coupling regime is assumed. In fact, we can do that if we disregard the last term $\sum_j\hbar\omega_j$ of Eq. (6). However, this term introduces an infinite sum of positive frequencies and comes from the quantization of the radiation field (see, e.g., page 11 of Walls and Milburn [6], and Section 2.6 of Milonni [15]). Thus, in this case, we can think that there are two competitive terms: the last one that introduces an infinite positive energy, and the others that introduce an infinite negative energy. It is a hard task to exactly compute the simultaneous contribution leading to a possible cancellation of these terms.

On the other hand, we would like to remind that the model of Eq. (1) does not have a lower bound when the oscillator frequency goes to zero [12].

3. The independent-oscillator (IO) model

The independent oscillator (IO) model [12,13] includes a self-interaction term, and is described by the Hamiltonian

$$\widehat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_j \hbar\omega_j\left(\hat{b}_j^\dagger\hat{b}_j + \frac{1}{2}\right) + \sum_j\left(\lambda_j\hat{b}_j + \lambda_j^*\hat{b}_j^\dagger\right)\left(\hat{a} + \hat{a}^\dagger\right) + \sum_j \frac{|\lambda_j|^2}{\hbar\omega_j}\left(\hat{a} + \hat{a}^\dagger\right)^2. \tag{7}$$

Through the same calculations of the foregoing section we find that, for a given α, the lower bound now results

$$\langle\Psi|\widehat{H}|\Psi\rangle\big|_{\min} = \hbar\omega_0\left(|\alpha|^2 + \frac{1}{2}\right) + \sum_j \hbar\omega_j. \tag{8}$$

Therefore, in this case there is no problem about the lower bound energy, although the problem concerning with an infinite positive energy persists. This is accomplished because of the inclusion of the self interaction term in the Hamiltonian of the IO model, a procedure not usual in the standard literature [4–9]. Where come from this term?

To answer this question, let us write this Hamiltonian in terms of $\hat{x}$ and $\hat{p}$ operators. In the case of the IO model the interaction considered is given by [12]:

$$\widehat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) + \sum_j\left(\frac{\hat{p}_j^2}{2m_j} + \frac{m_j\omega_j^2}{2}\hat{q}_j^2\right) - \hat{x}\sum_j m_j\omega_j\hat{q}_j + \sum_j \frac{m_j\omega_j^2}{2}\hat{x}^2, \tag{9}$$

and it is clear in this equation the inclusion of the self interaction term, given by $\hat{x}^2$. Indeed, the Eq. (7) is derived from Eq. (9). The inclusion of this term is due to consider that the single quantum system is independently coupled to each mode of the heat bath, that is to say, inside of the summation is the term $(\hat{q}_j - \hat{x})^2$.

On the other hand, the $\hat{x} - \hat{p}$ representation of the Hamiltonian given by Eq. (1) is:

$$\widehat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0}{2}\hat{x}^2 + \sum_j\left(\frac{\hat{p}_j^2}{2m_j} + \frac{m_j\omega_j^2}{2}\hat{q}_j^2\right) - \sqrt{m\omega_0}\,\hat{x}\sum_j\sqrt{m_j\omega_j}\,\lambda_j\hat{q}_j, \tag{10}$$

which is obtained by taking λ_j as a real constant and substituting in Eq. (1), as usual, the following definitions [12]:

$$\hat{a} = \frac{m\omega_0\hat{x} + i\hat{p}}{(2\hbar m\omega_0)^{1/2}}, \tag{11}$$

$$\hat{b}_j = \frac{m_j\omega_j\hat{q}_j + i\hat{p}_j}{(2\hbar m_j\omega_j)^{1/2}}. \tag{12}$$

At first sight, defining $\lambda_j = \sqrt{m_j\omega_j^3/(m\omega_0)}$ one can think the Eq. (10) as being derived from Eq. (9) by simply discarding the self interaction term. However, we can ask for the physical reason invoked in order to be in the possibility of doing this. The physical meaning for considering, in the interaction Hamiltonian, only the term $\hat{x}\sum_j m_j\omega_j\hat{q}_j$ is that the bath oscillators are attached to a fixed origin [12], which is not a good consideration, and inclusive, if it is valid, persist the problem of how to make the procedure of discarding the self interaction term. In fact, the only way to neglect it is by making $\hat{x} = 0$ inside the sum, but this means no interaction at all. Thus, we see that there are not good justification for going from Eq. (9) to Eq. (10). Therefore, we can conclude that the Hamiltonian (1), usually found in textbooks of quantum optics, is based on an unpractical assumption, never stated explicitly.

4. Conclusions

In this Letter we showed that the Hamiltonian usually discussed in textbooks of quantum optics to model a single quantum harmonic oscillator coupled to an infinite set of quantum harmonic oscillators, Eq. (1), has serious problems. In particular, it requires a valid explanation to discard the self interaction term. The answer to the question posed in the title: "Is there a lower bound energy in the harmonic oscillator interacting with a heat bath?", depends on the model Hamiltonian used. In the IO model the answer is yes, although this model suffers from the common infinities caused by the self interaction terms. In the non-rotating wave model, commonly found in textbooks of quantum optics, the answer is dependent on the complete cancellation of terms giving infinite negative energy (see the first and second term of Eq. (6)) and the self interaction term, which gives an infinite positive energy. In any case, we can draw the most important conclusion of this work: the Hamiltonian to be used in order to avoid the lower bound inconsistence is that of the independent oscillator model.

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