Influence of dissipation on parametric amplification of coupled atomic and optical fields

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In this paper, the influence of dissipation on the dynamics of a recently proposed model [Phys. Rev. A **60**, 1491 (1999)] of a strongly far-off-resonant pumped Bose-Einstein condensate interacting with a quantized light field is studied. It is verified that dissipation induces an exponential instability in the field mean intensities for values of parameters where the undamped system is stable. Also considered is the impact of dissipation on the quantum-statistical properties of the light and atomic fields.

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I. INTRODUCTION

The experimental realization of Bose-Einstein condensates (BECs) of dilute alkali-metal atoms [1,2] has stimulated research on the optical properties [3–6], statistical properties [7], phase properties [8–14], and tunneling effect [15,16] in atomic BECs, which has produced an intense development of the atom optics. BECs are now a current tool in the laboratories and have been used to produce physical processes that sometime ago were just subjects of theoretical studies. From the analogy between atom optics with conventional light optics, interesting physical phenomena like the atom lasers [17–20] and the matter-wave mixing [21,22] has been explored.

A topic of intense research in atom optics is the dynamics of a BEC interacting with single mode quantized light fields [23–31]. Opposite to the case where the light fields have a passive action on the atoms, in these works both matter and light fields are a dynamically coupled entity. Some results in this field have shown the possibility of controlling quantum statistical properties of atoms by using quantized light fields [23,25]. In this line recently developed by Moore *et al.* [26] was a linear model of interaction between a strongly pumped condensate and a quantized light field in a cavity. Such a system permits a simultaneous parametric amplification of coupled atomic and optical fields. The authors also demonstrated an optical control over the quantum statistics of matter waves by varying the light field initial intensity. In this model the effect of the dissipation in the cavity field was not taken into account. A system close to this one was studied by Law and Bigelow [24]. They have considered the amplification of an atomic beam crossing a heavy damped cavity filled by a BEC. The assumption of a very strong damped cavity was necessary in order to reach the matter wave amplification.

This paper is an extension of the model proposed in Ref. [26] by taking into account the dissipation in the cavity light field, turning the model more realistic and closer to current experiments [31]. The dissipation in the light field does not have precisely a detrimental effect, which could be expected in principle. It is verified that dissipation is responsible for a presence of a dynamical instability in conditions where the undamped system is stable. Furthermore, presented is a study of the role of dissipation in the system quantum statistical properties and its effects on optical control over the quantum statistics of the matter waves.

The paper is organized as follows. Section II presents a review of the model. In addition it takes into account dissipation on the cavity field. Section III discusses the dissipation effect on the system dynamics. Discussed is the role of dissipation on the system stability, its effects on the field mean intensities, and the impact of damping on the system quantum statistical properties. Section IV presents a summary and conclusion.

II. MODEL

The system considered here is an extension of a problem recently studied by Moore *et al.* [26]. It consists of a Schrödinger field of interacting bosonic two-level atoms with atomic mass m, atomic resonance frequency ω_a driven by a strong classical pump laser, and a weak quantized damped optical cavity mode (the probe mode), both being far-off resonant from atomic transition frequencies. Only collisions between ground-state atoms are considered. In the far-off-resonance regime, the excited-state population is negligible and spontaneous emission, center-of-mass motion, and collisions between excited atoms and between excited- and ground-state atoms, can be neglected. In this situation the excited state can be adiabatically eliminated resulting into an effective Hamiltonian between ground-state atoms and probe field, given by [26]

$$\hat{H}=\hbar ck\hat{A}^{\dagger}\hat{A}+\int d^3\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})\bigg[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r}) + \frac{2\pi\hbar^2\sigma}{m}\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})+\frac{\hbar}{2\Delta}[g\Omega_0^*\hat{A}e^{i(\omega_0 t+\mathbf{K}\cdot\mathbf{r})}+\text{H.c.}] + \frac{\hbar}{\Delta}\left(\frac{|\Omega_0|^2}{4}+g^2\hat{A}^{\dagger}\hat{A}\right)\bigg]\hat{\Psi}(\mathbf{r}). \quad (1)$$

In this Hamiltonian, Ω_0 is the Rabi frequency of the pump laser of frequency ω_0 and momentum $\mathbf{k}_0$, and $\hat{A}$ is the annihilation operator of the probe field of momentum $\mathbf{k}$ satisfying $[\hat{A},\hat{A}^{\dagger}]=1$. In the off-resonance approximation the index of refraction inside the condensate is negligible so that the probe mode frequency is $\omega\approx ck$, where k is the magnitude of vector $\mathbf{k}$. In addition, $\hat{\Psi}(\mathbf{r})$ is the atomic field operator for the ground-state atoms satisfying $[\hat{\Psi}(\mathbf{r}),\hat{\Psi}^\dagger(\mathbf{r}')]=\delta^3(r-r')$, and $V(\mathbf{r})$ is the trap potential for ground-state atoms. The vector $\mathbf{K}=\mathbf{k}-\mathbf{k}_0$ is the atomic recoil momentum resulting from absorption of a pump photon followed by emission of a probe photon and the parameter $\Delta=\omega_0-\omega_a$ is the detuning between pumping field and atomic transition frequency. The atom-probe coupling constant is $g=d[ck/2\hbar\epsilon_0 LS]^{1/2}$, where d is the magnitude of the atomic dipole moment, L is cavity length, and S is the cross-sectional area of the probe mode in the vicinity of the atomic sample, which is assumed to be approximately constant across the length of the atomic sample. Collisions between the ground-state atoms are taken into account in the s-wave approximation and characterized by the s-wave scattering length σ.

The motion equations for the condensate and probe mode can be obtained from the Hamiltonian (1), and are given by

$$\frac{d}{dt}\hat{\Psi}(\mathbf{r})=i\Bigg[\frac{\hbar}{2m}\nabla^2-\frac{V(\mathbf{r})}{\hbar}-\frac{4\pi\hbar\sigma}{m}\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) -\frac{g|\Omega_0|}{2|\Delta|}(\hat{a}e^{i\mathbf{K}\cdot\mathbf{r}}+\text{H.c.}) -\frac{1}{\Delta}\left(\frac{|\Omega_0|^2}{4}+g^2\hat{a}^\dagger\hat{a}\right)\Bigg]\hat{\Psi}(\mathbf{r}), \tag{2}$$

$$\frac{d}{dt}\hat{a}=(i\delta'-\kappa')\hat{a}-i\frac{g|\Omega_0|}{2\Delta}\int d^3\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\hat{\Psi}(\mathbf{r}) -i\frac{\Omega_0^*\Delta}{|\Omega_0||\Delta|}e^{i\omega_0 t}\hat{f}(t), \tag{3}$$

where $\delta'=\omega_0-\omega$ is the detuning between pump and probe fields and the operator $\hat{a}=-i\Omega_0^*\Delta/|\Omega_0||\Delta|e^{i\omega_0 t}\hat{A}$ was introduced for the probe mode. The dissipation in the probe mode was included in the Heisenberg equation of motion (3) by a cavity decay rate κ'. Also included was Langevin stochastic noise operators $\hat{f}(t)$ whose correlation functions at zero temperature are assumed to be Markovian and given by [32]

$$\langle\hat{f}^\dagger(t)\hat{f}(t')\rangle=0,\quad \langle\hat{f}(t)\hat{f}^\dagger(t')\rangle=2\kappa'\delta(t-t'). \tag{4}$$

It is assumed that the atomic field is initially a Bose-Einstein condensate, with a mean number of condensate atoms N, and that this condensate is well described by a number state so that the initial state of the atomic field may be taken as

$$|\psi(t=0)\rangle=\frac{1}{\sqrt{N!}}(c_0^\dagger)^N|0\rangle, \tag{5}$$

where $|0\rangle$ is the vacuum state, and

$$\hat{c}_0^\dagger=\int d^3\mathbf{r}\varphi_0(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}) \tag{6}$$

is the creation operator in the condensate state $\varphi_0(\mathbf{r})$, which satisfies the time-independent Gross-Pitaevskii equation

$$\left[\frac{\hbar}{2m}\nabla^2-\frac{V(\mathbf{r})}{\hbar}-\frac{4\pi\hbar\sigma}{m}N\left|\varphi_0(\mathbf{r})\right|^2+\frac{\mu}{\hbar}\right]\varphi_0(\mathbf{r})=0, \tag{7}$$

μ being the chemical potential. The motion equation for $\hat{c}_0$ is found differentiating (6) with respect to time and inserting Eqs. (7) and (2), which results in

$$\frac{d}{dt}\hat{c}_0=-i\left(\frac{\mu}{\hbar}+\frac{|\Omega_0|^2}{4\Delta}+\frac{g^2}{\Delta}\hat{a}^\dagger\hat{a}\right)\hat{c}_0+i\frac{4\pi\hbar\sigma}{m}\int d^3\mathbf{r}\varphi_0^*(\mathbf{r}) \times[N|\varphi_0(\mathbf{r})|^2-\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})]\hat{\Psi}(\mathbf{r}) -i\frac{g|\Omega_0|}{2|\Delta|}(\hat{a}\hat{c}_-+\hat{a}^\dagger\hat{c}_+), \tag{8}$$

where it was defined

$$\hat{c}_\pm=\int d^3\mathbf{r}\varphi_\pm(\mathbf{r})\hat{\Psi}(\mathbf{r}) \tag{9}$$

for the first-order condensate side modes with $\varphi_\pm(\mathbf{r})=\varphi_0(\mathbf{r})e^{\pm i\mathbf{K}\cdot\mathbf{r}}$.

A. Small-signal regime

Equation (8) shows that the role of optical fields is to couple the condensate mode to two side modes. In principle, the collision term in Eq. (8) also couples the condensate mode to various neighboring modes. However, to be consistent with the assumption of a pure condensate ($T=0$), it is assumed that collisions alone do not populate any different atomic states.

From here it is considered the small-signal regime. This means neglecting coupling with higher-order side modes than $\hat{c}_\pm$. In addition, it is neglected translational coupling as well as the term containing $\hat{a}^\dagger\hat{a}$ in Eq. (8), that is responsible for cross-phase modulation. These approximations are valid considering only short times so that the populations of side modes are small when compared with the population in the $\hat{c}_0$ mode and $\langle\hat{a}^\dagger\hat{a}\rangle\ll|\Omega|^2/4g^2$. With these considerations the model is simplified by expanding the atomic field operator as

$$\hat{\Psi}(\mathbf{r})\approx\varphi_0(\mathbf{r})\hat{c}_0+\varphi_-(\mathbf{r})\hat{c}_-+\varphi_+(\mathbf{r})\hat{c}_+. \tag{10}$$

The motion equation for the light field, maintaining the coupling between the three principal modes in the atomic operators, becomes

$$\frac{d}{dt}\hat{a}=(i\delta'-\kappa')\hat{a}-i\frac{g|\Omega_0|}{2|\Delta|}(\hat{c}_-^\dagger\hat{c}_0-\hat{c}_0^\dagger\hat{c}_+) -i\frac{\Omega_0^*\Delta}{|\Omega_0||\Delta|}e^{i\omega_0 t}\hat{f}(t). \tag{11}$$

The motion equation for $\hat{c}_0^\dagger\hat{c}_0$, $\hat{c}_-^\dagger\hat{c}_0$, and $\hat{c}_0^\dagger\hat{c}_+$ operators can be obtained from Eqs. (8), (9), and (10) which in the leading order in the collision and optical terms results in

$$\frac{d}{dt}\hat{c}_0^\dagger\hat{c}_0=i\frac{8\pi\hbar\sigma F_0}{m}\hat{c}_-^\dagger\hat{c}_0\hat{c}_+^\dagger\hat{c}_0+i\frac{g|\Omega_0|}{2|\Delta|}\hat{a}^\dagger(\hat{c}_-^\dagger\hat{c}_0-\hat{c}_0^\dagger\hat{c}_+)+\text{H.c.}, \tag{12}$$

$$\frac{d}{dt}\hat{c}_-^\dagger\hat{c}_0=i\omega_r\hat{c}_-^\dagger\hat{c}_0+i\frac{4\pi\hbar\sigma F_0}{m}(\hat{c}_-^\dagger\hat{c}_0+\hat{c}_0^\dagger\hat{c}_+)+i\frac{g|\Omega_0|^2}{2|\Delta|}\hat{a}\hat{c}_0^\dagger\hat{c}_0, \tag{13}$$

$$\frac{d}{dt}\hat{c}_0^\dagger\hat{c}_+=-i\omega_r\hat{c}_0^\dagger\hat{c}_+-i\frac{4\pi\hbar\sigma F_0}{m}(\hat{c}_-^\dagger\hat{c}_0+\hat{c}_0^\dagger\hat{c}_+)-i\frac{g|\Omega_0|^2}{2|\Delta|}\hat{a}\hat{c}_0^\dagger\hat{c}_0, \tag{14}$$

where,

$$F_0=\int d^3\mathbf{r}|\varphi_0(\mathbf{r})|^4 \tag{15}$$

and $\omega_r=\hbar K^2/2m$ is the atomic recoil frequency. The leading terms in Eqs. (11)–(14) were obtained by two principal considerations [26]. The first is Bose enhancement, which, in the regime of negligible condensate depletion, strongly selects transitions involving the condensate mode. The second is momentum conservation, which comes from the spatial integration in the polarization and collision terms. Integrals over slowly varying functions such as $|\varphi_0(\mathbf{r})|^2$ or $|\varphi_0(\mathbf{r})|^4$ are "momentum selected," and dominate over integrals of rapidly oscillating functions such as $|\varphi_0(\mathbf{r})|^2\exp(-\mathrm{i}\mathbf{K}\cdot\mathbf{r})$.

B. Linear response

In the small-signal regime a nonlinear system given by Eqs. (11)–(14) describing the coupling between the probe mode and three atomic modes was obtained. A further simplification is to consider that the condensate mode is initially occupied by a very large number N of atoms, which makes it possible to linearize the atomic field operators around their initial expectation values as

$$\hat{c}_0^\dagger\hat{c}_0=N(1+\hat{\delta}_0), \tag{16}$$

$$\hat{c}_-^\dagger\hat{c}_0=N\hat{\delta}_-, \tag{17}$$

and

$$\hat{c}_0^\dagger\hat{c}_+=N\hat{\delta}_+, \tag{18}$$

where $\hat{\delta}_0$, $\hat{\delta}_-$ e $\hat{\delta}_+$ are therefore infinitesimal operators. In addition, a rescaled probe field operator is introduced

$$\hat{\delta}_a=\frac{\hat{a}}{\sqrt{N}}, \tag{19}$$

which, provided that the mean number of photons in the probe mode is small compared to N, is also infinitesimal. Inserting these definitions into the motion Eqs. (11)–(14) and keeping only linear terms in the infinitesimal operators, the condensate mode can be eliminated from the dynamics and the problem is reduced to a 3×3 linear system of coupled differential equations between the probe mode and the atomic side modes, given by

$$\frac{d}{d\tau}\boldsymbol{\delta}=\mathrm{i}\tilde{\mathbf{M}}\boldsymbol{\delta}+\mathbf{F}(t), \tag{20}$$

with

$$\boldsymbol{\delta}=\begin{pmatrix}\hat{\delta}_a\\ \hat{\delta}_-\\ \hat{\delta}_+\end{pmatrix}, \tag{21}$$

$$\tilde{\mathbf{M}}=\begin{pmatrix}\delta+i\kappa & -\chi & -\chi\\ \chi & (1+\beta) & \beta\\ -\chi & -\beta & -(1+\beta)\end{pmatrix}, \tag{22}$$

and

$$\mathbf{F}(t)=\begin{pmatrix}\hat{F}(t)\\ 0\\ 0\end{pmatrix}, \tag{23}$$

where $\hat{F}(t)=-\mathrm{i}(\Omega_0^*\Delta e^{\mathrm{i}\omega_0 t}/|\Omega_0^*||\Delta|\omega_r\sqrt{N})\hat{f}(t)$ and the dimensionless time $\tau=\omega_r t$ together with the dimensionless parameters (in units of recoil frequency) was introduced

$$\chi=\frac{g|\Omega_0|\sqrt{N}}{2|\Delta|\omega_r}, \tag{24}$$

$$\delta=\delta'/\omega_r, \tag{25}$$

$$\beta=\frac{4\pi\hbar\sigma NF_0}{m\omega_r}, \tag{26}$$

and

$$\kappa=\kappa'/\omega_r, \tag{27}$$

where χ is the atom-probe coupling, δ is the pump-probe detuning, β is the strength of collisions between the side modes, and κ is the damping rate of the probe mode.

III. DYNAMICS

In this section the dynamical features of the interacting three-mode system obtained in the previous section is discussed. The solution of Eq. (20) can be formally written as

$$\boldsymbol{\delta}(\tau)=e^{i\tilde{\mathbf{M}}\tau}\boldsymbol{\delta}(0)+\int_0^\tau e^{i\tilde{\mathbf{M}}(\tau-t)}\mathbf{F}(t)dt, \tag{28}$$

which can be used to compute the temporal evolution of the field mean intensities as well as the higher-order correlations between the fields. The term containing the stochastic force in Eq. (28) does not contribute to the correlations involving normal ordering of operators because it was assumed a zero-temperature cavity [see Eq. (4)]. To solve the linear system it is necessary to find the eigenvalues and eigenvectors of the $\tilde{\mathbf{M}}$ matrix defined in Eq. (22). The characteristic frequencies of the system satisfy the cubic equation

$$\omega^3-(\delta+i\kappa)\omega^2-(1+2\beta)\omega+(1+2\beta)(\delta+i\kappa)+2\chi^2=0. \tag{29}$$

In the following a brief review about the system characteristics without losses is given and then the results of including dissipation into the probe mode are presented.

A. System without dissipation

In the absence of dissipation in the probe mode ($\kappa=0$) it was found by Moore *et al.* [26] that Eq. (29) has either three real solutions, or one real and a pair of complex conjugate solutions. In the first case, the system is stable and exhibits only small oscillations around its initial state. In the second case, the system is unstable and grows exponentially, even from the noise. The exponential instability occurs when

$$\chi^2>\frac{1}{27}[(3+6\beta+\delta^2)^{3/2}+\delta^3-9\delta(1+2\beta)]. \tag{30}$$

For $\chi^2\ll 1$ the region of instability is centered around $\delta=\sqrt{1+2\beta}$, which is consistent with the energy conservation in the scattering of a pump photon into the probe by an atom initially at rest. Equation (30) defines a threshold condition present in the system without dissipation.

B. Influence of dissipation

The complete solution of Eq. (29) can be found explicitly, but the analytical expression is complicated and does not provide so much insight. In order to obtain a transparent analytical result, let us first rewrite Eq. (29) in the following way:

$$(\omega-\overline{\omega}_1)(\omega-\overline{\omega}_2)(\omega-\overline{\omega}_3)+2\chi^2=0, \tag{31}$$

where $\overline{\omega}_1=\delta+i\kappa$, $\overline{\omega}_2=\sqrt{1+2\beta}$, and $\overline{\omega}_3=-\sqrt{1+2\beta}$ are the eigenfrequencies for the decoupled matter and light system. In principle it could be expected that the dissipation has just a detrimental effect in the system by damping the oscillations below the threshold and preventing the exponential growth above it in the coupled matter and light system. What is verified here is that the inclusion of dissipation induces an instability below the threshold and reduces the exponential growth above it. In order to explore the dissipation effect on the region below the threshold let us find a solution in first order in χ^2, writing the roots of Eq. (31) as

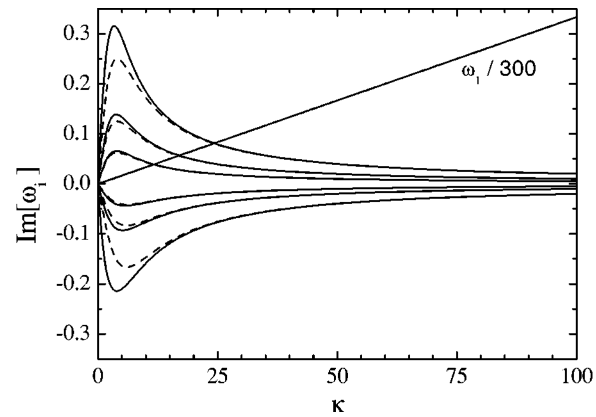

FIG. 1. Imaginary parts of the eigenfrequencies as a function of κ for $\delta=-5$, $\beta=0$ for $\chi^2=0.5$, 1.0, and 2.0. Dashed lines correspond to approximated results and solid lines to exact numerical values.

$$\omega_i=\overline{\omega}_i+\Delta\omega_i \quad i=1,2,3, \tag{32}$$

where $\Delta\omega_i$ are corrections in the free evolution eigenfrequencies due to the coupling parameter χ. Substituting Eq. (32) in Eq. (31) and keeping only the first-order terms in $\Delta\omega_i$, the following eigenfrequencies for the coupled system are found:

$$\omega_1=\delta-\frac{2\chi^2[\delta^2-(1+2\beta)-\kappa^2]}{[\delta^2-(1+2\beta)-\kappa^2]^2+4\delta^2\kappa^2}+i\kappa\left[1+\frac{4\delta\chi^2}{[\delta^2-(1+2\beta)-\kappa^2]^2+4\delta^2\kappa^2}\right], \tag{33}$$

$$\omega_2=\sqrt{1+2\beta}+\frac{\chi^2(\delta-\sqrt{1+2\beta})}{\sqrt{1+2\beta}[(\delta-\sqrt{1+2\beta})^2+\kappa^2]}-i\frac{\kappa\chi^2}{\sqrt{1+2\beta}[(\delta-\sqrt{1+2\beta})^2+\kappa^2]}, \tag{34}$$

$$\omega_3=-\sqrt{1+2\beta}-\frac{\chi^2(\delta+\sqrt{1+2\beta})}{\sqrt{1+2\beta}[(\delta+\sqrt{1+2\beta})^2+\kappa^2]}+i\frac{\kappa\chi^2}{\sqrt{1+2\beta}[(\delta+\sqrt{1+2\beta})^2+\kappa^2]}. \tag{35}$$

The eigenfrequencies ω_i (33)–(35) contain a frequency shift appearing in their real parts, that is due to the interaction between modes, present even in the absence of dissipation. Another point is that for nonzero dissipation ($\kappa\neq 0$) all the eigenfrequencies are complex. Note that for δ very close to $\pm\sqrt{1+2\beta}$ the results are not valid for small values of κ since the condition of small χ was assumed. In order to verify the accuracy of this approximation, Fig. 1 shows the behavior of the imaginary part of ω_1, ω_2, and ω_3 (dashed lines) as a function of κ comparing them with the exact solutions of the cubic Eq. (29) solved numerically (solid

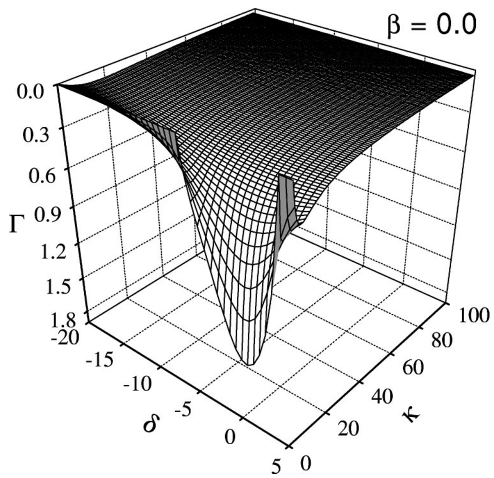

FIG. 2. Growth rate as a function of δ and κ for a coupling parameter $\chi^2=3$ without collisions.

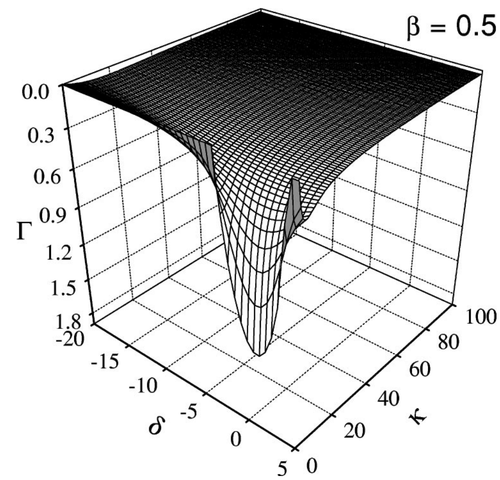

FIG. 3. Same as in Fig. 2 with inclusion of collisions.

lines) for several values of the coupling parameter χ^2. It is possible to see that the approximation is good for small values of χ^2 when compared with other parameters. At very strong dissipation ($\kappa \gg \chi^2, \delta, \sqrt{1+2\beta}$) the eigenfrequencies reduce to

$$\omega_1 \simeq \delta + \frac{2\chi^2}{\kappa^2} + \mathrm{i}\kappa, \tag{36}$$

$$\omega_2 \simeq \sqrt{1+2\beta} + \frac{\chi^2(\delta - \sqrt{1+2\beta})}{\kappa^2\sqrt{1+2\beta}} - i\frac{\chi^2}{\kappa\sqrt{1+2\beta}}, \tag{37}$$

and

$$\omega_3 \simeq -\sqrt{1+2\beta} - \frac{\chi^2(\delta + \sqrt{1+2\beta})}{\kappa^2\sqrt{1+2\beta}} + i\frac{\chi^2}{\kappa\sqrt{1+2\beta}}, \tag{38}$$

that are valid for all values of parameters irrespective if they lie below or above the condition given by Eq. (30).

In particular one of the roots (ω_2) has a negative imaginary part. This means that the damped system is unstable also in the region below the threshold defined in the ideal case without losses. At very large times the fields mean amplitudes will grow exponentially with a rate Γ given by the absolute value of imaginary part of ω_2

$$\Gamma = \frac{\kappa\chi^2}{\sqrt{1+2\beta}[(\delta - \sqrt{1+2\beta})^2 + \kappa^2]}, \tag{39}$$

that initially grows with the increase of κ, takes a maximum absolute value in $\kappa = |\delta - \sqrt{1+2\beta}|$, and decreases monotonically as

$$\frac{\chi^2}{\kappa\sqrt{1+2\beta}} \tag{40}$$

for very large values of κ. Furthermore, the value of Γ is asymmetrical for values of detuning δ, being larger for positive ones. Figure 2 shows the behavior of Γ, by solving numerically Eq. (29), as a function of κ and δ for fixed values of χ and β (3.0 and 0.0, respectively). It is verified that for the damped model there is no threshold condition. The model without dissipation corresponds, in the figure, to the $\kappa=0$ plane where the imaginary part is zero outside the region defined by Eq. (30). The effect of collisions is to introduce an extra reduction in the values assumed by Γ as can be seen in Fig. 3 for a collision parameter $\beta=0.5$. The collision parameter β lies in a range 0.1–1 for densities around 10^{15} cm^{-3} [26], and opposite to the dissipation, which is responsible by an instability in a region where the undamped system is stable, collisions contribute to a reduction in the rate of exponential growth.

C. Mean intensities

The field intensities $I_a = \hat{a}^\dagger\hat{a}$, $I_- = \hat{c}_-^\dagger\hat{c}_-$, and $I_+ = \hat{c}_+^\dagger\hat{c}_+$ are related to the $\hat{\delta}_j$ operators, with $j=a, -, +$, by

$$I_j(t) = N\hat{\delta}_j^\dagger(t)\hat{\delta}_j(t) - \delta_{j-}, \tag{41}$$

and their mean intensities $\overline{I_j}$ can be calculated using the solution (28) and the properties of the Langevin noise force. It is assumed that the side modes begin in the vacuum state, while the probe field is initially in a coherent state α. In this case the mean intensities can be given by

$$\overline{I_j}(\tau) = |G_{ja}(\tau)|^2|\alpha|^2 + |G_{j-}(\tau)|^2 - \delta_{j-}, \tag{42}$$

where

$$G_{ij}(\tau) = \sum_{k=1}^{3}[U]_{ik}[U^{-1}]_{kj}e^{\mathrm{i}\omega_k\tau} \tag{43}$$

and $[U]_{ik}$ is the ith component of the kth eigenvector of $\tilde{\mathbf{M}}$ matrix and ω_k the respective eigenvalue. The expression (42)

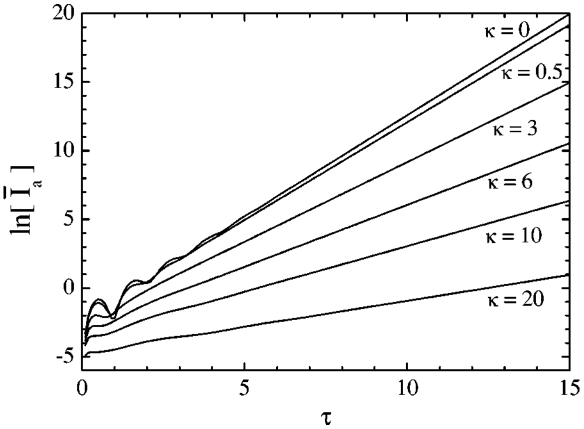

FIG. 4. Logarithmic plot of the probe mode mean intensity as a function of τ for $\delta=-5$, $\beta=0$, $\chi^2=4$ considering several values of damping rates.

contains a spontaneous contribution, present even in the case where $\alpha=0$, as well as a stimulated term proportional to the injected signal strength $|\alpha|^2$.

In the limit of very large time the eigenfrequency with a negative imaginary part, that we label $\omega_2=\Omega-i\Gamma$, will dominate in expression (43) and the fields are amplified growing exponentially. In such a limit Eq. (42) reduces to

$$\overline{I_j}(\tau)=(|b_{ja}\alpha|^2+|b_{j-}|^2)e^{2\Gamma\tau}, \tag{44}$$

where $b_{jk}=[U]_{j2}[U^{-1}]_{2k}$.

In order to demonstrate the dissipation effect in the intensities, let us consider just the probe mode for $\alpha=0$ and the absence of collision ($\beta=0$). Plotted in Fig. 4 is the logarithm of the probe mode intensity as a function of time for χ^2 above the threshold showing that the effect of dissipation is to reduce the rate of exponential growth when compared to the undamped case $\kappa=0$. Shown in Fig. 5 is the effect of dissipation in the probe mode intensity by considering a value of χ^2 below the threshold. It is seen that the intensity is an oscillating function in the absence of dissipation ($\kappa=0$). However, for $\kappa\neq 0$, after a transient, in which occurs damped oscillations, the probe mode is amplified and grows exponentially with a maximum growth occurring in $\kappa=|\delta-\sqrt{1+2\beta}|$ as predicted in Sec. III B.

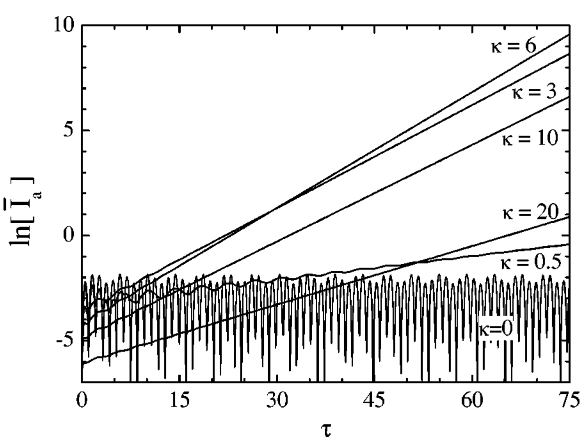

FIG. 5. Same as in Fig. 4 with $\chi^2=1$.

D. Quantum statistics

It was showed by Moore *et al.* [26] that for the system considered here without dissipation it is possible, above the threshold given by Eq. (30), to control the quantum statistical properties of the matter fields by varying the initial probe mode intensity. In this section the impact of the extra noise introduced by dissipation on the quantum-statistical properties of the system is studied. A possible way to characterize these properties is to work with the relative uncertainties in the field intensities that are given by

$$Q=\frac{\sqrt{\overline{(I_j-\overline{I_j})^2}}}{\overline{I_j}}. \tag{45}$$

For the initial conditions considered in the last section and using a normal ordering of operators it is found that

$$Q=\sqrt{1-\frac{|G_{ja}(\tau)\alpha|^4}{\overline{I_j^2}}+\frac{1}{\overline{I_j}}}, \tag{46}$$

which in the limit of very large times, where the solution (44) is valid, the relative uncertainty reduces to a stationary value

$$Q=\sqrt{1-\frac{|\alpha|^4}{[|\alpha|^2+f^2(\chi,\beta,\delta,\kappa)]^2}}, \tag{47}$$

where the following function was introduced:

$$f(\chi,\beta,\delta,\kappa)=\frac{|[U^{-1}]_{2-}|}{|[U^{-1}]_{2a}|}. \tag{48}$$

Note that expression (47) is not valid in the limit $\kappa=0$ for values of parameters not satisfying the condition (30), because in this limit the solutions are oscillating functions of time.

Equation (47) shows that by varying $|\alpha|$ it is possible to go continuously from a thermal ($Q=1$) to a coherent state ($Q=0$) and in this way control the atomic modes quantum statistical properties using the light field. Furthermore, it is expected that the extra noise introduced by the dissipation in the probe mode produces a diffusive process increasing the values of the function $f(\chi,\beta,\delta,\kappa)$ leading the modes close to a thermal state. The effect of the dissipation in the modes quantum statistics is showed in Figs. 6 and 7. Figure 6 shows the behavior of Q to values of χ and δ in the region where the undamped system is stable (excluding the $\kappa=0$ plane). Figure 7 is the same as Fig. 6 but with a value of δ that lies in the region where the system is unstable even in the undamped case. It can be seen in these figures that for small values of damping rate it is possible to vary the quantum statistical with a small variation of $|\alpha|$. However, as soon as the damping rate reaches some recoil frequencies the diffusive process induced by the dissipation brings the system to

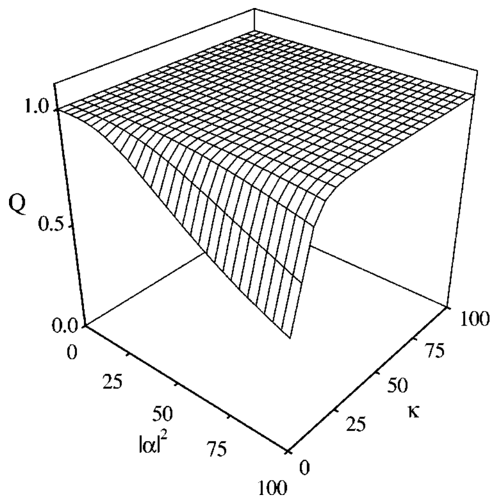

FIG. 6. Relative uncertainties as a function of $|\alpha|^2$ and κ for $\delta=-5$, $\beta=0$, and $\chi^2=1$.

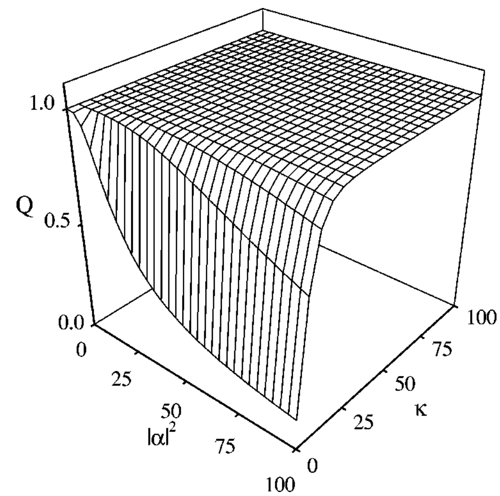

FIG. 7. Same as in Fig. 6 with $\delta=1$.

a thermal state. These results show that in real cavities it is necessary to have a larger range of variation for the initial intensity $|\alpha|$ of the probe field in order to get some control of the atomic mode quantum statistics.

IV. SUMMARY AND CONCLUSION

In this paper, the dissipation effects in the simultaneous parametric amplification of matter and light fields was studied. The inclusion of dissipation induces an instability in a region of parameters where the undamped system is stable. This instability induced by the dissipation can be physically understood observing that for the system without damping, the range of frequency in which the photons can be stronger scattered into the probe mode is centered in $\delta=\sqrt{1+2\beta}$ (for small coupling between light and atomic fields). The damping spreads this range by 2κ as $\sqrt{1+2\beta}-\kappa<\delta<\sqrt{1+2\beta}+\kappa$. Then the probability of occurrence of photons bunching in the probe mode and the rescattering of atoms to side atomic modes and so on increases even at small values of coupling parameter χ. A competition between excitation and losses is clearly expected and the last one will dominate when large values are attained.

Also studied was the impact of the dissipation in the quantum statistical properties of the system. The extra noise introduced by the dissipation brings the system close to a thermal state, a large variation of injected light field intensity being necessary in order to produce appreciable changes in the atomic fields quantum statistical properties. Using a pumped cavity can be a possible way to overcome this detrimental effect over the controlling of quantum statistical properties and this will be considered in a future work.

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